The structure of finite Morse index solutions to two free boundary problems in $\mathbb{R}^2$

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Abstract

We give a description of the structure of finite Morse index solutions to two free boundary problems in $\mathbb{R}^2$. These free boundary problems are models of phase transition and they are closely related to minimal hypersurfaces. We show that these finite Morse index solutions have finite ends and they converge exponentially to these ends at infinity. As an important tool in the proof, a quadratic decay estimate for the curvature of free boundaries is established.

Keywords: Finite Morse index solution; phase transition; free boundary problem; minimal surface.

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Contents

1 Introduction

In nonlinear elliptic problems, the structure of stable or finite Morse index solutions is always of great interest. For example, the classification of stable minimal hypercones is related to Bernstein problem and it plays an important role in the partial regularity theory for minimal hypersurfaces. For sets with minimal perimeter (i.e. minimal hypersurfaces which are *minimizers*) in $\mathbb{R}^n$, the celebrated Bernstein theorem states that if $n \leq 7$, it must be flat, i.e. an half space. For *stable* minimal hypersurfaces, it has been long conjectured that the same should be true, although only the dimension 3 case was proved (M. do Carmo and C.-K. Peng [9], Fischer-Colbrie and R. Schoen [18]). It turns out that this characterization of stable minimal surfaces in $\mathbb{R}^3$ and various tools developed in its proof (e.g. interior curvature estimates for stable minimal surfaces [41]) is very helpful in the study of the structure of minimal surfaces in $\mathbb{R}^3$, e.g. in the Colding-Minicozzi theory.

Although by now only a few results about stable minimal hypersurfaces are known in high dimensions, it was proved by Cao et. al. [6] that, for any n, a stable minimal hypersurface in $\mathbb{R}^n$ has one end (in other words, it is connected at infinity). For minimal hypersurfaces with finite Morse index, Li and Wang [32] also show the finteness of ends. In $\mathbb{R}^3$ this was known for a long time, because a minimal surface with finite Morse index in $\mathbb{R}^3$ has finite total curvature (Fishcer-Colbrie [17]), and a classical result of Osserman says such a surface is conformal to a Riemannian surface with finitely many points removed (corresponding to the ends), see for example [34, Section 2.3].

In the realm of the Allen-Cahn equation,

$$\Delta u = u^3 - u, \tag{1.1}$$

we face a similar situation. Due to its close connection with minimal hypersurfaces, there is the De Giorgi conjecture corresponding to the Bernstein problem, concerning the one dimensional symmetry of entire solutions. Similar to the minimal surface theory, for *minimizers* of (1.1), the De Giorgi conjecture has been proved by Savin [40] (see also the author [46] for a new proof). For *stable* solutions, at present the one dimensional symmetry is only known to be true in dimension 2 (an observation of Dancer). The *stable De Giorgi conjecture* in $\mathbb{R}^{n-1}$ also implies the original De Giorgi conjecture in $\mathbb{R}^n$, see [2].

In view of results in the minimal surface theory, it is conjectured that (see for example [12] and [21])

Conjecture *A finite Morse index solution of* (1.1) *in* $\mathbb{R}^2$ *has finitely many ends.*

This resembles the structure theory for minimal surfaces in $\mathbb{R}^3$ with finite Morse index. Such a theory is possible because we already have a characterization of stable solutions, i.e. the stable Bernstein / De Giorgi result in dimension 3 and 2.

Here an end of u is defined to be an unbounded connected component of $\{u = 0\}$. In fact, this conjecture says that near infinity, a finite Morse index solution is composed by a finite number of stable solution (which is one dimensional by the *stable De Giorgi conjecture*), patched together suitably.

It turns out that at present this conjecture is still not proven. As far as the author knows, up to now the only known results in this direction is concerned with solutions with finite ends, see [21, 10, 29, 30, 31].

In this paper, we study two free boundary problems related to the Allen-Cahn equation and prove the above conjecture for these two problems.

1.1 The first problem

The first problem is

$$\begin{cases} \Delta u = 0, & \text{in } \Omega := \{-1 < u < 1\}, \\ u = \pm 1, & \text{outside } \Omega, \\ |\nabla u| = 1, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Throughout this paper we only consider classical solutions, i.e. Ω is assumed to be an open domain of $\mathbb{R}^2$ with smooth boundary and $u \in C(\mathbb{R}^2) \cap C^2(\overline{\Omega})$, with the equation and the boundary conditions in (1.2) satisfied pointwisely. Of course, in view of the regularity theory for free boundaries in [1] and [48], this hypothesis can be relaxed a lot, but we will not pursue it here.

Equation (1.2) arises as the Euler-Langarange equation of the functional

$$\int |\nabla u|^2 + \chi_{\{-1<u<1\}}. \tag{1.3}$$

Note that $\chi_{\{-1<u<1\}}$ can be viewed as a double well potential, although in a rather degenerate manner.

This problem has been studied in [3] and [27]. In [3], Caffarelli and Córdoba put this problem in a continuous family of phase transition models with double well potentials and proved the uniform $C^{1,\alpha}$ regularity of Lipschitz transition layers in the corresponding singular perturbation problems. Compared to the Allen-Cahn equation, their treatment is only different in some technical points. In [27], Kamburov developed the techniques introduced in [13] which deals with the Allen-Cahn equation, and gave a counterexample of the De Giorgi conjecture for (1.2) in $\mathbb{R}^n$, $n \geq 9$.

Define the quadratic form

$$Q(\eta) := \int_\Omega |\nabla \eta|^2 - \int_{\partial\Omega} H\eta^2, \quad \eta \in C_0^\infty(\mathbb{R}^2),$$

where H is the mean curvature of $\partial\Omega$ with respect to ν, the unit normal vector of $\partial\Omega$ pointing to Ω^c. This is the second variation form of the functional (1.3) at u, for the derivation see [4] and [26].

A solution u is said to be of finite Morse index, if

$$\sup \dim\{X : X \text{ subspace of } C_0^\infty(\mathbb{R}^2), Q\lfloor_X \le 0\} < +\infty.$$

A standard argument shows that a finite Morse index solution is stable outside a compact set, that is, there exists an $R_0 > 0$ such that, for any $\eta \in C_0^\infty(\mathbb{R}^2 \setminus B_{R_0}(0))$,

$$\int_\Omega |\nabla \eta|^2 \ge \int_{\partial\Omega} H\eta^2. \tag{1.4}$$

Our main result for this problem can be stated as follows:

Theorem 1.1. *Let u be a solution of (1.2) in $\mathbb{R}^2$. Assume Ω to be connected. If u is stable outside a compact set, then*

1. *u satisfies the natural energy growth bound*

$$\int_{B_R(0)} \frac{1}{2}|\nabla u|^2 + \chi_{\{-1<u<1\}} \le CR,$$

for some constant C (depending on u) and all $R > 0$;

2. *the total curvature is finite,*

$$\int_\Omega |\nabla^2 u|^2 - |\nabla|\nabla u||^2 < +\infty.$$

3. *for some $R > 0$ large, there are only finitely many connected components of $\Omega \setminus B_R$, which we denote by D_i, $1 \le i \le N$ for some $N > 0$;*

4. *for some $R > 0$ large, $\Omega^c \setminus B_R$ consists only of unbounded connected components, which is finite;*

5. *each D_i has the form*

$$D_i = \{x : f_i^-(e_i \cdot x + b_i) \le e_i^\perp \cdot x + a_i \le f_i^+(e \cdot x + b_i)\},$$

where e_i is a unit vector, a_i and b_i are constants, and $f_i^\pm$ two smooth functions defined on $[R, +\infty)$, f_i^+ convex and f_i^- concave, satisfying

$$f_i^- < f_i^+ < f_i^- + 4;$$

6. *we have the balancing formula*

$$\sum_{i=1}^{N} e_i = 0;$$

7. *for each i, there exists a constant κ_i so that both the limits $\lim_{t\to+\infty}\left(f_i^{\pm}(t)-\kappa_i t\right)$ exist, where the convergence rate is exponential, and*

$$\lim_{t\to+\infty}\left[f_i^{+}(t)-f_i^{-}(t)\right]=2.$$

An important result used in the proof of this theorem is the following characterization of stable solutions.

Theorem 1.2. *Let u be a stable solution of* (1.2)*. Then there exists a unit vector e such that $u(x)\equiv u(x\cdot e)$.*

In Theorem 1.1, each end D_i corresponds to a stable solution. As in the Allen-Cahn equation [19] and [2], the proof of Theorem 1.2 uses the Liouville theorem for the degenerate elliptic equation

$$\operatorname{div}(\sigma^2\nabla\psi)=0. \tag{1.5}$$

Another method to prove Theroem 1.2 involves a geometric Poincaré inequality of Sternberg-Zumbrun type, see for example [43] and [16]. The following quantity appears in this inequality,

$$|A|^2 := \frac{|\nabla^2 u|^2-|\nabla|\nabla u||^2}{|\nabla u|^2} = |B|^2+|\nabla_T\log|\nabla u||^2,$$

where B is the second fundamental form of the level set (the curvature of the level set because we are in $\mathbb{R}^2$) and ∇_T is the tangential derivative along the level set. This geometric Poincaré inequality is also used in this paper to establish a local integral curvature bound. In turn, this curvature bound implies that at infinity the solution is close to a one dimensional solution at $O(1)$ scale.

In Theorem 1.1, if the hypothesis on the connectedness of Ω is removed, we have the following strong half space theorem, which holds for any solution of (1.2), without any stablility condition.

Theorem 1.3. *Let u be a solution of* (1.2) *in $\mathbb{R}^2$. If Ω is not connected, then u is one dimensional.*

Note that we can single out the restriction of u to a connected component of Ω as a solution to (1.2). Hence two components of Ω give two such solutions, u_1 and u_2, satisfying $u_1\ge u_2$. This is similar to the situation met in the strong half space theorem for minimal surfaces [23]. Of course, compared to their proofs, the proof of our strong half space theorem is rather direct. This is because the free boundary $\partial\Omega$ is convex (mean convex if we are in higher dimension spaces).

1.2 The second problem

The second problem is a one phase free boundary problem,

$$\begin{cases} \Delta u = W'(u), \quad \text{in } \Omega := \{u > 0\}, \\ u = 0, \quad \text{outside } \Omega, \\ |\nabla u| = \sqrt{2W(0)}, \quad \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

The solution is a critical point of the following functional

$$\int \frac{1}{2}|\nabla u|^2 + W(u)\chi_{\{u>0\}}. \tag{1.7}$$

Here W is a standard double well potential, that is, $W \in C^2[0, +\infty)$ satisfying

W1) *$W \geq 0$, $W(1) = 0$ and $W > 0$ in $[0, 1)$;*

W2) *for some $\gamma \in (0, 1)$, $W' < 0$ on $(\gamma, 1)$;*

W3) *there exists a constant $\kappa > 0$, $W'' \geq \kappa > 0$ on $[\gamma, +\infty)$;*

W4) *there exists a constant $p > 1$, $W'(u) \geq c(u-1)^p$ for $u > 1$.*

A typical example is $W(u) = (1-u^2)^2/4$ which gives the standard Allen-Cahn nonlinearity.

The potential $W(u)\chi_{\{u>0\}}$ can still be viewed as a double well potential, degenerate on the negative side. The one phase free boundary problem, especially the partial regularity theory for its free boundaries, has been studied for a long time, see for example [1, 48]. This problem also arises in the study of Serrin's overdetermined problem, see [47], where a De Giorgi type conjecture was proved for minimizers of (1.7) in $\mathbb{R}^n$, $n \leq 7$.

The finite Morse index condition can be defined similarly as in problem (1.2). This condition still implies that u is stable outside a compact set, that is, there exists an $R_0 > 0$ such that, for any $\eta \in C_0^\infty(\mathbb{R}^2 \setminus B_{R_0}(0))$,

$$\int_\Omega |\nabla \eta|^2 + W''(u)\eta^2 \geq \int_{\partial\Omega} \left(-\frac{W'(0)}{\sqrt{2W(0)}} + H \right) \eta^2. \tag{1.8}$$

Here H is the mean curvature of $\partial\Omega$ with respect to ν, the unit normal vector of $\partial\Omega$ pointing to Ω^c.

Our main result for this problem is similar to the first one.

Theorem 1.4. *Let u be a solution of (1.6) in $\mathbb{R}^2$. Assume Ω to be connected. If u is stable outside a compact set, then*

1. *u satisfies the natural energy growth bound*

$$\int_{B_R(0)} \frac{1}{2}|\nabla u|^2 + W(u)\chi_{\{u>0\}} \leq CR,$$

for some constant C (depending on u) and all $R > 0$;

2. *the total curvature is finite,*

$$\int_{\Omega} |\nabla^2 u|^2 - |\nabla|\nabla u||^2 < +\infty.$$

3. *for any $R > 0$ large, there are only finitely many connected components of $\Omega \setminus B_R$;*

4. *for some $R > 0$ large, $\Omega^c \setminus B_R$ consists only of finitely many unbounded connected components, which we denote by D_i, $1 \leq i \leq N$ for some $N > 0$;*

5. *each D_i has the form*

$$D_i = \{x : f_i^-(e_i \cdot x + b_i) \leq e_i^\perp \cdot x + a_i \leq f_i^+(e \cdot x + b_i),$$

where e_i is a unit vector, a_i and b_i are constants, and $f_i^\pm$ two smooth functions defined on $[R, +\infty)$, f_i^+ concave and f_i^- convex;

6. *for each i, both the limits*

$$\kappa_i^\pm := \lim_{t\to+\infty} \frac{df_i^\pm}{dt}(t)$$

exist. Moreover, by denoting $e_i^\pm$ the asymptotic direction of the curve $\{e_i^\perp \cdot x + a_i = f_i^\pm(e \cdot x + b_i)\}$ at infinity, we have the balancing formula

$$\sum_{i=1}^{N} \left(e_i^+ + e_i^-\right) = 0;$$

7. *if $i \neq j$, $\{e_i^+, e_i^-\} \cap \{e_j^+, e_j^-\} = \emptyset$;*

8. *the limits*

$$\lim_{t\to+\infty} \left(f_i^\pm(t) - \kappa_i^\pm t\right)$$

exist, where the convergence rate is exponential.

Compared to Theorem 1.1, the new point is (7). This is because in the positive part $\{u > 0\}$, different ends are pushed away, while in the first problem, a pair of $\partial\{u = 1\}$ and $\partial\{u = -1\}$ stay at finite distance. For these two problems, outside Ω, different ends could be at finite distance (for example, we could have $\kappa_i^+ = \kappa_i^-$ in Theorem 1.4 (6)), because in this part there is no interaction between different ends.

With this description, we can prove

Corollary 1.5. *Let u be a solution of (1.6) satisfying all of the hypothesis in Theorem 1.4. Assume furthermore that u has only two ends, then it is one dimensional.*

Since the number of ends is even, this implies that u has at least four ends, unless it is one dimensional.

As in the first problem, an important result used in the proof of this theorem is the following characterization of stable solutions. The proof is similar to the one for Theorem 1.2.

Theorem 1.6. *Let u be a stable solution of (1.6). Then there exists a unit vector e such that $u(x) \equiv u(x \cdot e)$.*

Similar to Theorem 1.3, we also have a strong half space theorem for (1.6).

Theorem 1.7. *Let u be a solution of (1.6) in $\mathbb{R}^2$. If Ω is not connected, then u is one dimensional.*

This can be proved by the same method as in Theorem 1.3.

1.3 Idea of the proof

Although these two free boundary problems look more complicated than the Allen-Cahn equation, it turns out that the classification of finite Morse index solutions to these two problems is a little simpler than the Allen-Cahn equation. This is mainly due to the convexity (mean convexity if the dimension larger than 2) of free boundaries in these two problems. This convexity is a consequence of the Modica type inequality. Although it is also believed that the Modica inequality in the Allen-Cahn equation gives a kind of (mean) convexity, it seems not so easy to realize this. For a connection of the Modica inequality with the convexity in the Allen-Cahn equation, see [42].

This convexity provides us with the crucial Lipschitz regularity of free boundaries at infinity. This Lipschitz regularity, combined with the stability of the solution near infinity and some topological considerations, then allows us to deduce the finiteness of ends. For the first problem, this is fairly direct, because by using the stability we can show that at infinity a pair of $\partial\{u = 1\}$ and $\partial\{u = -1\}$ stay at finite distance. (This fact also

implies that bounded connected components of Ω^c stay in a fixed compact set.) For the second problem, because different components of $\partial\Omega$ are expected to be pushed away on the positive side, we have to make use of an idea of Dancer in [8], using the stability condition to prove that the number of nodal domains of solutions to the linearized equation of (1.6) is finite. This uses the Liouville property for nonnegative subsolutions to the degenerate elliptic equation (1.5). Of course, the dimension 2 is crucial here. Using the convexity of free boundaries, this finiteness information is transferred to u, which implies the finiteness of unbounded components of Ω.

In the second problem, the above argument does not give any information on bounded components of Ω^c. To show that bounded components of Ω^c stay in a fixed compact set, we turn to a quadratic curvature decay, which can be stated as

$$H(x) \le \frac{C}{1+|x|}, \quad \text{on } \partial\Omega.$$

This type of decay estimate for curvatures, e.g. Schoen's curvature estimate [41], has been used a lot in the study of minimal surfaces. However, it is not so direct to derive such a decay estimate in our setting. This is possibly due to the lack of a Simons type equality for the curvature form $|A|^2$ in semilinear elliptic equations. Hence an indirect approach is taken in this paper. First, we use the blow up method (more precisely, the doubling lemma of Polácik-Quittner-Souplet [37]) to reduce the estimate to some uniform estimates in the corresponding singular perturbation problem, that is:

Question (CD): *Consider a solution u_ε to the problem*

$$\begin{cases} \varepsilon \Delta u_\varepsilon = \dfrac{1}{\varepsilon} W'(u_\varepsilon), & \textit{in } \{u_\varepsilon > 0\} \cap B_1(0), \\ |\nabla u_\varepsilon| = \dfrac{1}{\varepsilon}\sqrt{2W(0)}, & \textit{on } \partial\{u_\varepsilon > 0\} \cap B_1(0). \end{cases}$$

Assume the energy of u_ε is uniformly bounded and $\partial\{u_\varepsilon > 0\}$ is bounded in $C^{1,1}$ norm. Can we deduce that the curvature of $\partial\{u_\varepsilon > 0\}$ converges to 0 uniformly?

Stated in this this way, the answer to this question would be yes if we can establish a uniform $C^{2,\alpha}$ estimate for $\partial\{u_\varepsilon > 0\}$.

Due to the presence of free boundaries, under the hypothesis in Question (CD), $\{u_\varepsilon > 0\}$ can have two forms:

- Case 1. (Multiplicity 1) $\{x_2 > f_\varepsilon(x_1)\}$ for a concave function f_ε;
- Case 2. (Multiplicity 2) $\{f_\varepsilon^-(x_1) < x_2 < f_\varepsilon^+(x_1)\}$ for a concave function f_ε^- and a convex function f_ε^+.

Our proof of the above curvature estimate still uses the blow up/harmonic approximation method of De Giorgi, that is, we perform a blow up along the vertical direction and shows that the blow up limit is a harmonic (linear) function cf. Wickramasekera [50, 51]. Of course, the fact that we work in dimension 2 and free boundaries are all convex allow us to bypass many difficulties. But as we will see, there appear many new difficulties not encountered in minimal surface theory, mainly due to the fact that we are working with semilinear elliptic equations.

As can be expected, the multiplicity 1 case is comparatively simple. It can be proved by the first variation argument, i.e. substituting suitable test functions into the stationary condition. This is possible because various error terms appearing in this procedure is exponentially small.

It takes more effort to deal with the second case. This can be compared with the regularity theory for codimension 1 stationary varifolds, cf. [50, 51]. Because the multiplicity is 2, the blow up limit could be:

- **Subcase 2.1.** $\tilde{f}^+ \equiv \tilde{f}^-$;
- **Subcase 2.2.** $\tilde{f}^+ - \tilde{f}^- < +\infty$ in an interval and $\tilde{f}^+ - \tilde{f}^- = +\infty$ in other places;
- **Subcase 2.3.** $\tilde{f}^+ - \tilde{f}^- < +\infty$ everywhere but it is not identically 0;
- **Subcase 2.4.** $\tilde{f}^+ - \tilde{f}^- = +\infty$ everywhere.

The first two cases can be excluded by a direct first variation argument. In this paper, we mainly use the k-th order momentum

$$\int_{f_\varepsilon^-(x_1)}^{f_\varepsilon^+(x_1)} x_2^k \left[\frac{1}{\varepsilon}W(u_\varepsilon) + \frac{\varepsilon}{2}\Big|\frac{\partial u_\varepsilon}{\partial x_2}\Big|^2 - \frac{\varepsilon}{2}\Big|\frac{\partial u_\varepsilon}{\partial x_1}\Big|^2\right], \quad k = 0, 1, 2, \cdots.$$

This is equivalent to the first variation argument.

In Subcase 2.3, first we need to show that $\tilde{f}^\pm$ are linear when they are apart, i.e. when $\tilde{f}^+ > \tilde{f}^-$. This problem is almost trivial in minimal surface theory, cf. [51]. But it is not the case in semilinear elliptic equations and our proof of this fact is quite involved. It uses the Hamiltonian inequality of Gui [20], an estimate on the error between the solution and canonical one dimensional solutions. (More details will be discussed when we come to the exponential convergence of the solution at infinity.) After establishing this piecewise linearity, the next task is to exclude the possibility of contact points. In other words, we want to show that $\tilde{f}^+ > \tilde{f}^-$ everywhere. This is achieved by choosing a nontrivial intermediate scale and perform the blow up analysis at this scale. The choice of this intermediate scale makes use of the continuity from $O(1)$ scale to $O(\varepsilon)$ scale in singular perturbation problems.

In Subcase 2.4, we expect the interaction between the two components $\{x_2 = f_\varepsilon^\pm(x_1)\}$ to be negligible, and each component behaves like the multiplicity 1 case. However, it seems not so easy to estimate the interaction between these two components. Instead, a sliding method similar to [40, Section 5] is employed, which shows that the blow up limit is linear in the viscosity sense.

It should be emphasized that, in Case 1 and Subcase 2.4, the oscillation of $f_\varepsilon^\pm$ could be of the order $O(\varepsilon)$. Even in this case, we still get an improvement of decay estimate. Using the language in [46], this says that even if the excess is of the order $O(\varepsilon^2)$, an improvement of tilt-excess estimate could be still possible. It seems that the nature of this improvement estimate is different from the one in [46]. Our proof is still variational, using the first variation argument. This is different from the viscosity method in [40].

Another fact we want to call the readers' attention is, the stability condition is not needed in the above proof of Question (CD). (It is only used when we applying the doubling lemma to reduce the quadratic curvature decay to Question (CD).) If we add the stability condition, based on the calculation in [11], it is natural to conjecture that the above Case 2 cannot occur. In other words, for stable solutions, the distance between different components of free boundaries (or transition layers) has a uniform positive lower bound. The author has a method to give a weaker estimate: the distance is bounded from below by $\varepsilon|\log\varepsilon|$. The uniform positive lower bound is not known yet.

Next, let us discus the exponential convergence of u at infinity. This is relatively independent of other results in this paper. Once we know that at infinity u is close to a finite number of one dimensional solutions patched together and some uniform Lipschitz regularity of $\partial\Omega$, we can further show that the convergence rate (to the one dimensional profile) is exponential. This is mainly due to the following two facts: (i) because we are in dimension 2, the minimal hypersurfaces are just straight lines, and hence there is no effect of the curvature; (ii) the second eigenvalue for the linearized problem at the one dimensional solution g is positive. Note that the one dimensional solution is stable. However, there is an eigenfunction with eigenvalue 0. Fortunately this eigenfunction is exactly g', which comes from the translation invariance of the problem.

To prove the exponential convergence, we view the equation as an evolution problem in the form

$$\frac{d^2u}{dt^2} = \nabla\mathcal{J}(u),$$

where $\mathcal{J}$ is the corresponding functional defined on the real line $\mathbb{R}$. Let $\mathcal{M}$ be the manifold of one dimensional solutions. (This mainfold is the real line $\mathbb{R}$, formed by translations of a one dimensional solution g.) Take the nearest point $P(u)$ on $\mathcal{M}$ to u. Then roughly speaking, $u - P(u)$ almost lie in the subspace orthogonal to the first eigenfunction of $P(u)$.

By some more computations we get

$$\frac{d^2}{dt^2}\|u - P(u)\|^2 \geq \mu\|u - P(u)\|^2 + \mathcal{R}, \tag{1.9}$$

where μ is a positive constant (related to the second eigenvalue of g). The norm $\|\cdot\|$ is usually taken to be a L^2 one. The remainder term $\mathcal{R}$ is of the order $O(e^{-ct})$ for some constant $c > 0$. This then implies the exponential convergence of $\|u - P(u)\|^2$, and the exponential convergence of $u(t)$ with some more work.

This approach was used in Gui [20]. In this paper we take a related but different one. For the first problem, we use the L^2 norm of $1 - |\nabla u|$ to control the convergence rate. This quantity is in fact equivalent to the L^2 norm of u_t. For the second problem, we take $P(u)$ to be the one dimensional solution with the same free boundary point. This turns out to be a rather good approximation of the nearest point and it is sufficient for our use.

As we have mentioned above, to deal with Subcase 2.3 in the proof of Question (CD), we need to show that the blow up limit is piecewise linear. This was achieved by establishing an estimate on the L^2 norm of the difference between $u_\varepsilon(x_1, \cdot)$ and a canonical one dimensional solution, by adapting Gui's approach to the multiplicity 2 case. In Gui's original approach (and also in the above exponential convergence problem), in the Hamiltonian identity the energy difference between u_ε and the one dimensional solution is exponentially small. This makes various error terms appearing in this approach integrable. However, this is not the case any more in Question (CD), because now we are encountered with a local problem. An estimate on the second order derivatives is introduced to overcome this difficulty. Finally, using the terminology in [51], what we have used in this method corresponds to the (height) coarse excess. The object in our problem corresponding to the finer excess used in [51] need the introduction of Fermi coordinates as in [13]. This could give a more precise estimate, but the coarse one is sufficient for our purpose.

Another approach to prove the exponential convergence using linear operator theory in weighted Sobolev spaces is presented in del Pino-Kowalczyk-Pacard [10]. However, due to the presence of free boundaries, it is not obvious to extend this method to our setting. A second problem related to this linear theory is that, in this paper we do not show the equivalence of the condition of stability outside a compact set and the finite Morse index condition. (The Schrödinger operator case was proved in [14].) We do not give any relation between the Morse index and the number of ends either. For related discussion about the Morse index of minimal surfaces, see for examples [7] and [33].

We also would like to mention that the positivity of the second eigenvalue can be viewed as a *nondegeneracy* result, because the first eigenvalue 0 comes from the translation invariance of the problem. This fact has been used a lot in the construction of solutions to the Allen-Cahn equation, see for example [11].

Finally, in a recent preprint of Jerison and Kamburov [25], they also study the structure of solutions to one phase free boundary problems in $\mathbb{R}^2$, and this paper is in fact inspired

by theirs. Their study is more on the line of the Colding-Minicozzi theory, that is, instead of the finite Morse index condition, they add some assumptions on the topology of the set $\Omega = \{u > 0\}$ and from these assumptions the structure of solutions is obtained. For Serrin's overdetermined problem, the topology and geometry of solutions in $\mathbb{R}^2$ have also been studied in [22, 38, 39, 45], which could be more complicated due to the lack of a variational structure. Here we want to emphasize that the stability condition is very strong and it provides much better control than these conditions. One example is the curvature estimates derived from the stability condition, which gives us a clear picture of the solution at $O(1)$ scale and enables us to obtain the convergence of translations of a fixed solution.

The paper is divided into three parts, the first two dealing with the two problems separately. The organization of the first two parts are almost the same and can be read independently. We first prove some uniform estimates for entire solutions of these two free boundary problems such as the Modica inequality. Then we prove the one dimensional symmetry of stable solutions and use this and an integral curvature estimate to prove the finiteness of ends. In the last step we prove the refined asymptotics at infinity. In Part II, more effort is needed to prove the finiteness of bounded connected components of Ω^c, the proof of which is postponed to Part III due to its length. The whole Part III is devoted to the proof of Theorem 9.1.

Part I
The first problem

In dimension 1, the problem (1.2) has a solution g defined by

$$g(x) = \begin{cases} 1, & x \geq 1, \\ x, & -1 < x < 1, \\ -1, & x \leq -1. \end{cases}$$

Of course, the trivial extension $u(x_1, x_2) := g(x_1)$ is a solution of (1.2). Furthermore, for any $\cdots < t_i < t_i + 2 \leq t_{i+1} < t_{i+1} + 2 \leq \cdots$, with $i \in I$ (I finite or countably infinite), the function

$$v^*(x_1, x_2) := \sum_{i \in I} \left[(-1)^{i-1} \chi_{\{t_{i-1}+1 \leq x_1 \leq t_i - 1\}} + (-1)^i g(x_1 - t_i) \right]$$

is still a solution of (1.2). Moreover, it is stable in $\mathbb{R}^2$. We call such solutions one dimensional.

Notation: $\partial^{\pm}\Omega = \partial\Omega \cap \{u = \pm 1\}$.

2 Uniform estimates

In this section we prove a Modica type inequality and establishes the convexity of free boundaries. Note that we do not need any stability condition here and u only denotes a solution to (1.2).

The main result in this section is

Proposition 2.1. *In Ω, $|\nabla u| \le 1$.*

We would like to interprete this gradient bound as a Modica inequality. As the following proof shows, this gradient bound holds for entire solutions of (1.2) in $\mathbb{R}^n$, for any $n \ge 1$.

To prove Proposition 2.1, we need the following two lemmas, which are basically consequences of the Hopf Lemma.

Lemma 2.2. *There exists a constant d_A such that,*

$$\mathit{dist}(x, \partial\Omega) \ge d_A$$

for any $x \in \{-3/4 < u < 3/4\}$.

Proof. Take an arbitrary point $x \in \{-3/4 < u < 3/4\}$. Assume $\mathrm{dist}(x, \partial\Omega) =: h$ is attained at $y \in \partial^-\Omega$. Then $B_h(x)$ is tangent to $\partial^-\Omega$ at y.

The function $\tilde{u} := 1 + u$ is non-negative, harmonic in $B_h(x)$, satisfying $\tilde{u}(x) > 1/4$ and $\tilde{u}(y) = 0$. Hence, by the Hopf lemma, there exists a universal constant c such that

$$1 = |\nabla\tilde{u}(y)| = \left|\frac{y-x}{|y-x|} \cdot \nabla\tilde{u}(y)\right| \ge \frac{c\tilde{u}(x)}{h} = \frac{2c}{h}. \tag{2.1}$$

Hence $h \ge 2c$. □

Lemma 2.3. *There exists a universal constant C such that $|\nabla u| \le C$ in Ω.*

Proof. For $x \in \Omega \backslash \{x : \mathrm{dist}(x, \partial\Omega) > d_A/4\}$, $B_{d_A/8}(x) \subset \Omega$. Applying the standard gradient estimate for harmonic functions we deduce that

$$|\nabla u(x)| \le \frac{C}{d_A} \sup_{B_{d_A/8}(x)} |u| \le C.$$

Next, given $x \in \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) \le d_A/4\}$, let $y \in \partial\Omega$ attain $\mathrm{dist}(x, \partial\Omega) =: r$. Without loss of generality, assume $y \in \partial^-\Omega$. Then $\tilde{u} := 1 + u$ is positive, harmonic in $B_r(x)$. Since $y \in \partial B_r(x)$ and $\tilde{u}(y) = 0$, as in (2.1), we deduce that

$$\tilde{u}(x) \le Cr.$$

Then by the Harnack inequality,

$$cr \leq \inf_{B_{r/2}(x)} \tilde{u} \leq \sup_{B_{r/2}(x)} \tilde{u} \leq Cr.$$

By the interior gradient estimate for harmonic functions,

$$|\nabla u(x)| = |\nabla \tilde{u}(x)| \leq C \frac{\operatorname{osc}_{B_{r/2}(x)} \tilde{u}}{r} \leq C. \qquad \square$$

Now we come to the proof of Proposition 2.1.

Proof of Proposition 2.1. Take $x_k \in \Omega$ so that $|\nabla u(x_k)| \to \sup_\Omega |\nabla u|$, which we assume to be strictly larger than 1. The proof is divided into two cases.

Case 1. $\operatorname{dist}(x_k, \partial\Omega)$ does not converges to 0.
Consider

$$u_k(x) := u(x_k + x).$$

Clearly u_k is still a solution of (1.2) in $\mathbb{R}^2$. Since they are uniformly bounded in $\operatorname{Lip}(\mathbb{R}^2)$, we can assume that u_k converges to u_∞ uniformly on any compact set of $\mathbb{R}^2$.

The set $\Omega_\infty := \{-1 < u_\infty < 1\}$ is open. It can be directly checked that $\Delta u_\infty = 0$ in Ω_∞.

By the above construction, the origin $0 \in \Omega_\infty$ and

$$\lambda := |\nabla u_\infty(0)| = \sup_{\Omega_\infty} |\nabla u_\infty| > 1.$$

Since $\Delta |\nabla u_\infty|^2 = 2|\nabla^2 u_\infty|^2 \geq 0$ in the connected component of Ω_∞ containing 0, by the strong maximum principle, $|\nabla u_\infty|$ is constant and $\nabla^2 u_\infty \equiv 0$ in this component. Note that we still have $|u_\infty| \leq 1$ in $\mathbb{R}^n$. Hence, after a rotation and a translation, this component is $\{|x_1| < 1/\lambda\}$ and in this domain

$$u_\infty(x) \equiv \lambda x_2.$$

However, arguing as in the proof of [25, Lemma 4.2] we can also show that $\lambda \leq 1$. (Roughly speaking, this is because u_k are classical solutions, hence their uniform limit u_∞ is a viscosity subsolution.) This is a contradiction and we finish the proof in this case.

Case 2. $h_k := \operatorname{dist}(x_k, \partial\Omega) \to 0$.
Take a point $y_k \in \partial\Omega$ to attain this distance. Without loss of generality, assume $y_k \in \partial^-\Omega$.

Let

$$v_k(x) := \frac{1}{h_k} \left[1 + u_k(y_k + h_k x) \right].$$

By Lemma 2.2, $u < -3/4$ in $B_{2h_k}(y_k)$. Thus in $B_2(0)$, v_k is nonnegative and harmonic in its positivity set. Hence it is subharmonic in $B_2(0)$.

Denote $(x_k - y_k)/h_k$ by z_k. Since $|z_k| = 1$, we can assume it converges to z_∞.

As in Case 1, v_k converges to a Lipschitz function v_∞ uniformly on any compact set, and $\Delta v_\infty = 0$ in $\{v_\infty > 0\}$.

Since $\Delta v_k = 0$ and $v_k > 0$ in $B_1(z_k)$, by noting that $|\nabla v_k(z_k)| > 1$, we obtain

$$\inf_{B_r(z_k)} v_k \geq c(r) > 0, \quad \text{for } \forall r \in (0,1) \text{ and some constant } c(r),$$

where $c(r)$ is independent of k. Thus $\{v_\infty > 0\}$ is nonempty.

Still as in Case 1, we get

$$|\nabla v_\infty(z_\infty)| = \sup_{\{v_\infty > 0\}} |\nabla v_\infty| > 1.$$

Using the strong maximum principle, after suitable rotation and translation, v_∞ has the form $|\nabla v_\infty(z_\infty)| x_2^+$. This leads to a contradiction with the free boundary condition as in Case 1. $\square$

Proposition 2.4. *Every connected component of $\{u = 1\}$ or $\{u = -1\}$ is convex. Moreover, it is strictly convex unless u is one dimensional.*

This follows from the calculation in [4] (see also [25] and [26]).

This convexity implies that

Corollary 2.5. *Ω is unbounded.*

Proof. Assume $\Omega \subset B_R(0)$ for some $R > 0$. Then the connected component of Ω^c containing $\mathbb{R}^2 \setminus B_R(0)$ is the whole $\mathbb{R}^2$, because it is convex. This is clearly a contradiction. $\square$

Now we come to the proof of Theorem 1.3. Thus assume Ω is not connected and take two different connected components of Ω, Ω_i, $i = 1, 2$. For each $i = 1, 2$, define u_i to be the restriction of u to Ω_i, with the obvious extension to Ω_i^c. Then u_i are still solutions of (1.2).

Since Ω_2 is connected, it is contained in a connected component of Ω_1^c, say D. Because D is convex, it is contained in an half space H, say $\{x_1 > 0\}$. In this setting, we have the following weak half space theorem.

Theorem 2.6. *Let v be a solution of (1.2). Assume that $\{-1 < v < 1\}$ is contained in an half space. Then v is one dimensional.*

Proof. As before, we can assume $\{-1 < v < 1\}$ to be connected and it is contained in $\{x_1 > 0\}$.

Denote by D the connected component of $\mathbb{R}^2 \setminus \{-1 < v < 1\}$ containing $\{x_1 < 0\}$. Since D is convex, it can be directly checked that ∂D is a graph in the form $\{x_1 = f(x_2)\}$. By definition, f is a nonnegative concave function, hence a constant. Then Proposition 2.4 implies that g is one dimensional. $\square$

Theorem 1.3 follows from this theorem.

For applications below, we present a non-degeneracy result for the set $\{u = \pm 1\}$.

Proposition 2.7. *If $u \neq v^*$ (with a unit vector e and $\cdots < t_i \leq t_{i+1} \leq \cdots$, where there are two constants $t_i = t_{i+1}$), then for every $x \in \partial^\pm \Omega$ and $r > 0$, $|B_r(x) \setminus \Omega| > 0$.*

Proof. Assume there is an $x_0 \in \partial\Omega$ and $r_0 > 0$, such that $|B_{r_0}(x_0) \setminus \Omega| = 0$. By the convexity of $\partial\Omega$, $\partial\Omega \cap B_{r_0}(x_0)$ are straight lines. By Lemma 2.4, in $B_{r_0}(x_0)$, $u = v^*$ for a unit vector e and two constants $t_1 = t_2$. Then by Theorem 1.3 and the unique continuation principle applied to u, this holds everywhere in $\mathbb{R}^2$. □

In the following, we will always assume u satisfies this non-degeneracy condition.

3 The stable De Giorgi conjecture and its consequences

In this section we prove the stable De Giorgi conjecture. Then we use the stability to derive an integral curvature bound and use this to study the convergence of translations at infinity of a solution u to (1.2), which is assumed to be stable outside a compact set.

We first prove Theorem 1.2.

Proof. Similar to [26, Section 2.3], the stability condition implies the existence of a positive function $\varphi \in C^\infty(\overline{\Omega})$, satisfying

$$\begin{cases} \Delta\varphi = 0, & \text{in } \Omega, \\ \varphi_\nu = -H\varphi, & \text{on } \partial\Omega. \end{cases}$$

By direct differentiation, for any unit vector e, the directional derivative u_e also satisfies this equation.

Let $\psi := u_e/\varphi$. It satisfies

$$\begin{cases} \operatorname{div}\left(\varphi^2 \nabla\psi\right) = 0, & \text{in } \Omega, \\ \psi_\nu = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

For any $\eta \in C_0^\infty(\mathbb{R}^2)$, testing this equation with $\psi\eta^2$ and integrating by parts on Ω, we obtain

$$\int_\Omega \varphi^2 |\nabla\psi|^2 \eta^2 + 2\varphi^2 \eta\psi \nabla\eta\nabla\psi = 0.$$

By the Cauchy inequality,

$$\int_\Omega \varphi^2 |\nabla\psi|^2 \eta^2 \leq 8 \int_\Omega \varphi^2 \psi^2 |\nabla\eta|^2.$$

Then we can use standard log cut-off test functions to show that

$$\int_\Omega \varphi^2 |\nabla \psi|^2 = 0.$$

As in [19] or [2], this implies the one dimensional symmetry of u. $\square$

In the following part of this section, we assume u is a solution of (1.2), stable outside a compact set of $\mathbb{R}^2$.

Lemma 3.1. *For any $L > 1$, there exists an $R(L)$ such that, there is no bounded component of $\{u = \pm 1\}$ contained in $B_{R(L)}(0)^c$ with diameter smaller than L.*

Proof. **Step 1.** For any R large, take a $\varphi \in C_0^\infty(\mathbb{R}^2 \setminus B_R)$ and test the stability condition with $\varphi|\nabla u|$. After some integration by parts, we obtain

$$\begin{aligned}\int_{\partial\Omega} H\varphi^2 &\leq \int_\Omega |\nabla\varphi|^2|\nabla u|^2 + 2\varphi|\nabla u|\nabla\varphi\cdot\nabla|\nabla u| + \varphi^2|\nabla|\nabla u||^2 \\ &= \int_{\partial\Omega} \frac{1}{2}\varphi^2\left(|\nabla u|^2\right)_\nu + \int_\Omega |\nabla\varphi|^2|\nabla u|^2 - \varphi^2|A|^2|\nabla u|^2. \end{aligned} \tag{3.2}$$

On $\partial\Omega$,

$$\left(|\nabla u|^2\right)_\nu = 2u_{\nu\nu} = -2H.$$

Hence (3.2) is transformed to

$$\int_\Omega |\nabla u|^2|A|^2\varphi^2 \leq \int_\Omega |\nabla u|^2|\nabla\varphi|^2.$$

Then we can use standard log cut-off test functions to show that

$$\int_{\Omega\setminus B_R} |\nabla u|^2|A|^2 \leq \frac{C}{\log R},$$

which converges to 0 as $R \to +\infty$.

Step 2. For any $\eta \in C_0^\infty(B_R(0)^c)$,

$$\begin{aligned}\int_{\partial\{u>0\}} H\eta &= -\int_{\partial\{u>0\}} \left(\frac{|\nabla u|^2}{2}\right)_\nu \eta \\ &= -\int_{\{u>0\}} \nabla\frac{|\nabla u|^2}{2}\nabla\eta + \Delta\frac{|\nabla u|^2}{2}\eta \\ &= \int_{\{u>0\}} -\nabla^2 u\nabla u\nabla\eta + \eta|\nabla^2 u|^2. \end{aligned} \tag{3.3}$$

Claim. In $\{u>0\}$,
$$|\nabla^2 u \nabla u|^2 \leq |\nabla u|^2 |A|^2, \tag{3.4}$$
and
$$|\nabla^2 u|^2 = 2|\nabla u|^2 |A|^2. \tag{3.5}$$
This can be proved by writing these quantities in the coordinate form and using the equation $\Delta u = 0$.

Assume there is a connected component of $\{u=0\}$, D, contained in $B_R(0)^c$ with its diameter smaller than L. Take a point x in this component and η to be a standard cut-off function in $B_{2L}(x)$ with $\eta \equiv 1$ in $B_L(x)$. Substituting this into (3.3) and using (3.4), (3.5), noting that $H \geq 0$ on $\partial\Omega$, we obtain
$$\begin{aligned}\int_{\partial D} H &\leq \frac{C}{L}\int_{B_{2L(x)}} |\nabla u||A| + C\int_{B_{2L(x)}} |\nabla u|^2|A|^2 \\ &\leq C\int_{B_{2L(x)}} |\nabla u|^2|A|^2 + C\left(\int_{B_{2L(x)}} |\nabla u|^2|A|^2\right)^{\frac{1}{2}} \\ &\leq \frac{C}{\sqrt{\log R}}.\end{aligned}$$

On the other hand, because D is convex, the Gauss-Bonnet theorem says
$$\int_{\partial D} H = 2\pi.$$
Thus we get a contradiction if
$$\frac{C}{\sqrt{\log R}} < 2\pi. \qquad \square$$

As a corollary, we have

Corollary 3.2. *For any $x \in \partial\Omega \setminus B_{R(L)}(0)$ and $r \in (0, L/2)$, the connected component of $\partial\Omega \cap B_r(x)$ passing through x, denoted by $\Gamma^{x,r}$, has its boundary in $\partial B_r(x)$. Hence,*
$$\mathcal{H}^1(\Gamma^{x,r}) \geq 2r.$$

The proof of Lemma 3.1 in fact gives

Corollary 3.3. *For any $\varepsilon > 0$ small and $L > 0$ large, there exists an $R(L,\varepsilon)$ so that the following holds. For any $x \in \partial\Omega \setminus B_{R(L,\varepsilon)}(0)$, the connected component of $\partial\Omega$ passing through x, $\Gamma^{x,L}$, satisfies*
$$\int_{\Gamma^{x,L}\cap B_L(x)} H \leq \varepsilon,$$

and

$$dist_H(\Gamma^{x,L} \cap B_L(x), \{e^{x,L} \cdot (y-x) = 0\} \cap B_L(x)) \leq \varepsilon,$$

where $e^{x,L}$ *is a unit vector.*

Next we claim that

Lemma 3.4. *For any* $\ell \geq 1$ *and* $x_k \in \Omega$, $|x_k| \to \infty$, *the translation function*

$$u_k(x) := u(x_k + x)$$

converges in the C^ℓ *sense to* $v^*(e \cdot x)$ *for some unit vector* e *and a sequence of*

$$\cdots < t_i < t_i + 2 \leq t_{i+1} < t_{i+1} + 2 \leq \cdots, \quad i \in I.$$

Moreover, the translations of Ω, $\Omega_k := \Omega - x_k$ *converges to* $\{-1 < v^*(e \cdot x) < 1\}$ *in the* C^ℓ *sense on any compact set.*

In the above, we say Ω_k converges in the C^ℓ sense, if $\partial\Omega_k$ can be represented by the graph of a family of functions defined on $\{e \cdot x = 0\}$, with these functions converge to $\{e \cdot x = t_i\}$ respectively in C^ℓ. Note that this implies, for k large, Ω_k is C^ℓ diffeomorphic to $\{-1 < v^*(e \cdot x) < 1\}$. We say u_k converges to $v^*(e \cdot x)$ in the C^ℓ sense, if the push back of u_k through the above diffeomorphism converges to $v^*(e \cdot x)$ in $C^\ell_{loc}(\overline{\{-1 < v^*(e \cdot x) < 1\}})$.

Proof. First assume u_k converges to a limit function u_∞ uniformly on any compact set of $\mathbb{R}^2$. Of course $\Delta u_\infty = 0$ in $\{-1 < u_\infty < 1\}$.

By the previous corollary, $\partial\{-1 < u_k < 1\}$ converges in the Hausdorff distance to a family of lines, which we assume to be parallel to the x_1-axis.

For any k large, take a connected component D_k of $\{-1 < u_k < 1\}$. By the analysis above, ∂D_k consists of two convex curves $\Gamma_{1,k}$ and $\Gamma_{2,k}$. Moreover, there exist two constants $t_{1,k}$ and $t_{2,k}$ such that,

$$\lim_{k \to +\infty} \mathrm{dist}_H(\Gamma_{i,k}, \{x_2 = t_{i,k}\}) = 0, \quad i = 1, 2. \tag{3.6}$$

After a translation in the x_2 direction, we can assume $\overline{D_k}$ converges to $\overline{D_\infty} = \{0 \leq x_2 \leq t_\infty\}$ in the Huasdorff distance, where

$$t_\infty = \lim_{k \to +\infty} (t_{2,k} - t_{1,,k}) \in [0, +\infty].$$

Next we divide the proof into two cases.

Case 1. Assume $u_k = -1$ on $\Gamma_{1,k}$ and 1 on $\Gamma_{2,k}$.

By Lemma 2.2, $|t_{2,k} - t_{1,k}| \geq 2d_A$. This implies

$$t_\infty \geq 2d_A. \tag{3.7}$$

Take an arbitrary point $x_k \in \Gamma_{1,k}$. Then the following function is well defined in $B_{d_A}(x_k)$:

$$v_k(x) := \begin{cases} 0, & x \in B_{d_A}(0) \setminus (D_k - x_k), \\ u_k + 1, & x \in D_k - x_k. \end{cases}$$

Indeed, $(\Gamma_{1,k} - x_k) \cap B_{d_A}(0)$ is a convex curve with boundary points in $\partial B_{d_A}(0)$, hence it divides $B_{d_A}(0)$ into two connected open sets, one is $D_k - x_k$ where $v_k > 0$ and the other one $B_{d_A}(0) \setminus (D_k - x_k)$ where $v_k = 0$.

In $B_{d_A}(0)$, v_k is a classical solution of the one phase free boundary problem

$$\begin{cases} \Delta v_k = 0, & \text{in } \{v_k > 0\}, \\ |\nabla v_k| = 1, & \text{on } \partial\{v_k > 0\}. \end{cases} \tag{3.8}$$

(3.6) implies that $\partial\{v_k > 0\}$ are flat in the sense of [1]. Hence by the regularity theory for free boundaries in [1] and the higher regularity of free boundaries in [28], $\partial\{v_k > 0\} \cap B_{d_A/2}(0)$ can be represented by the graph of a function f_k defined on the x_1-axis, with its C^ℓ norm uniformly bounded for any $\ell \geq 1$.

This implies, for any $\ell \geq 1$, ∂D_k converges to ∂D_∞ in C^ℓ on any compact set. Then by standard elliptic estimates, u_k are uniformly bounded in $C^\ell_{loc}(\overline{D_k})$ for any $\ell \geq 1$, and they converge to u_∞ in the C^ℓ sense.

Now u_∞ satisfies

$$\begin{cases} \Delta u_\infty = 0, & \text{in } \{0 < x_2 < t_\infty\}, \\ -1 < u_\infty < 1, & \text{in } \{0 < x_2 < t_\infty\}, \\ u_\infty = -1, & \text{on } \{x_2 = 0\}, \\ u_\infty = 1, & \text{on } \{x_2 = t_\infty\}, \\ |\nabla v_\infty| = 1, & \text{on } \{x_2 = 0\} \cup \{x_2 = t_\infty\}. \end{cases}$$

In the above, if $t_\infty = +\infty$, we understand that the boundary condition on $\{x_2 = t_\infty\}$ is void.

We claim that $u_\infty = -1 + x_2$ in D_∞, and hence $t_\infty = 2$. This can be proved by many methods. A direct way is by noting that we have $|\nabla u_\infty| \leq 1$ in D_∞ (obtained by passing to the limit in $|\nabla u_k| \leq 1$), hence we can use Proposition 2.4 to deduce that $\nabla^2 u_\infty \equiv 0$ in D_∞ and the claim follows.

Case 2. Assume $u_k = -1$ on ∂D_k.

As in Case 1, the following function is well defined:

$$v_k(x) := \begin{cases} 0, & \text{outside } D_k, \\ u_k + 1, & \text{in } D_k. \end{cases}$$

v_k is a classical solution of the one phase free boundary problem (3.8).

This case can be further divided into two subcases.

Subcase 1. $\lim_{k\to 0} |t_{k,2} - t_{k,1}| = 0$.

Because $|\nabla v_k| \leq 1$ and $v_k = 0$ on $\partial\{v_k > 0\}$, we have

$$\sup_{D_k} v_k \to 0.$$

Take a standard cut-off function $\eta \in C_0^\infty(B_2(0))$ with $\eta = 1$ in $B_1(0)$. Then by (3.8), integrating by parts gives

$$\begin{aligned} \int_{\partial D_k} \eta &= -\int_{D_k} v_k \Delta\eta \\ &\leq \left(\sup_{D_k} v_k\right) \int_{D_k} |\Delta\eta| \to 0. \end{aligned}$$

On the other hand, because ∂D_k are convex curves satisfying (3.6),

$$\lim_{k\to+\infty} \mathcal{H}^1(\partial D_k \cap B_1(0)) = 4.$$

This is a contradiction.

Subcase 2. $\lim_{k\to 0} |t_{k,2} - t_{k,1}| \in (0, +\infty]$.

As in Case 1, u_∞ satisfies

$$\begin{cases} \Delta u_\infty = 0, & \text{in } \{0 < x_2 < t_\infty\}, \\ -1 < u_\infty < 1, & \text{in } \{0 < x_2 < t_\infty\}, \\ u_\infty = -1, & \text{on } \{x_2 = 0\} \cup \{x_2 = t_\infty\}, \\ |\nabla v_\infty| = 1, & \text{on } \{x_2 = 0\} \cup \{x_2 = t_\infty\}. \end{cases}$$

In the above, if $t_\infty = +\infty$, we understand that the boundary condition on $\{x_2 = t_\infty\}$ is void.

Still as in Case 1, by applying Proposition 2.4, $\nabla^2 u_\infty = 0$ in D_∞, which then leads to a contradiction. Thus this case does not appear. □

4 Finite ends

In this section, u still denotes a solution of (1.2), stable outside a compact set of $\mathbb{R}^2$. By the results established in the previous section, we can take a $K \gg 1$ and then $R_1 \gg 1$ so that, for any $x \in \Omega \setminus B_{R_1}$,

(F1) $|\nabla u| \geq 1 - 1/K$ in $\Omega \cap B_K(x)$;

(F2) $\Omega \cap B_K(x) = \cup_i \Upsilon_i$, where

$$\Upsilon_i = \{x : f_i^-(x \cdot e) < x \cdot e^\perp < f_i^+(x \cdot e)\},$$

with e a unit vector and

$$\cdots < f_{i-1}^+ < f_i^- < f_i^+ < \cdots ;$$

(F3) $f_i^\pm$ are defined on $(-K, K)$, with their C^ℓ norm bounded for any $\ell \geq 1$, uniformly in x;

(F4) each f_i^- is concave and each f_i^+ is convex;

(F5) $2 - 1/K \leq f_i^+ - f_i^- \leq 2 + 1/K$ in $(-K, K)$.

With these preliminaries we prove

Lemma 4.1. *Each connected component of Ω^c intersects $B_{R_1+2K}(0)$.*

Proof. Let D be an arbitrary connected component of Ω^c. Assume it does not intersect $B_{R_1+2K}(0)$. Without loss of generality, assume $u = 1$ in D.

Since D is a convex set, ∂D is a simple curve or two disjoint simple curves. (These curves could be unbounded.) If the latter case happens, $\mathbb{R}^2 \setminus D$ has two connected components. Then the component of $\mathbb{R}^2 \setminus D$ which does not intersect $B_{R_1+2K}(0)$ contains a component of Ω and it does not intersect Ω. This is clearly a contradiction with our assumption that Ω is connected. Hence ∂D is a single simple curve.

By our preliminary analysis, for each $x \in \partial D$, there exists a unit vector $e(x)$ such that,

$$\partial D \cap B_K(x) = \{y : y \cdot e^\perp = f^+(y \cdot e)\}.$$

Moreover, there exists a curve

$$\Gamma = \{y : y \cdot e^\perp = f^-(y \cdot e)\},$$

where $2 - 1/K \leq f^+ - f^- \leq 2 + 1/K$, such that

$$\{y \in B_K(x) : f^-(y \cdot e) < y \cdot e^\perp < f^+(y \cdot e)\} \subset \Omega.$$

In fact, this curves can be represented in the form

$$y + t(y)\nu(y), \tag{4.1}$$

where the function $t(y)$ is defined on $\partial D \cap B_K(x)$ and $\nu(y)$ is the unit normal of ∂D at y, pointing to Ω.

By abusing notations, we denote the connected component of $\partial\Omega$ where Γ lies on, still by Γ. Note that Γ is also a simple smooth curve.

By continuation, any point $x \in \Gamma$ satisfies $\mathrm{dist}(x, \partial D) \leq 2 + 1/K$. Hence Γ dose not intersect $B_{R_1+K}(0)$. Γ can still be represented by the graph of a function defined on ∂D as in (4.1). Γ and ∂D bounds a connected component of Ω, which does not intersect $B_{R_1+K}(0)$ either. However, this is a contradiction with our hypothesis that Ω is connected and the proof is completed. $\square$

This lemma can be reformulated as follows: every connected component of $\partial\Omega$ intersects $B_{R_1+2K}(0)$. Together with the facts (F1-5), this implies that there are only finitely many unbounded components of $\partial\Omega$. Because the above proof also implies that, for any unbounded component of Ω^c, its boundary is a simple smooth curve, we obtain

Corollary 4.2. *There are only finitely many unbounded connected components of* Ω^c.

Checking the proof of Lemma 4.1, we also obtain

Corollary 4.3. *There is no bounded connected component of* Ω^c *belonging to* $\mathbb{R}^2 \setminus B_{R_1+2K}(0)$.

This implies that $\partial\Omega \setminus B_{R_1+2K}(0)$ is composed by finitely many unbounded simple curves. Hence we have

Corollary 4.4. *There are only finitely many connected components of* $\Omega \setminus B_{R_1+2K}(0)$.

Putting all of the above facts together we get the following picture.

Lemma 4.5. *There exists a constant $R_2 > R_1 + 2K$ so that the following holds. Let $\Omega \setminus B_{R_2}(0) = \cup_{i=1}^N D_i$, where each D_i is connected. For every i, there exist a unit vector e_i, two constants a_i and b_i, and two bounded functions $f_i^{\pm}$ defined on $[R_2, +\infty)$, f_i^+ convex and f_i^- concave,*

$$f_i^- \leq f_i^+ \leq f_i^- + 2 + 1/K,$$

such that

$$D_i = \{x : f_i^-(e_i \cdot x + b_i) \leq e_i^{\perp} \cdot x + a_i \leq f_i^+(e \cdot x + b_i).$$

Note that we do not claim e_i to be different for different i.

Now we come to the natural energy growth bound.

Lemma 4.6. *There exists a constant C depending on u, such that, for any $R > 1$,*

$$\int_{B_R} |\nabla u|^2 + \chi_\Omega \leq CR.$$

Proof. In view of Proposition 2.1, we only need to prove

$$|\Omega \cap B_R| \leq CR. \tag{4.2}$$

This follows directly from the previous lemma. □

For each $\varepsilon > 0$, let

$$u_\varepsilon(x) := u(\varepsilon^{-1}x).$$

Proposition 4.7. *As $\varepsilon \to 0$,*

$$\varepsilon|\nabla u_\varepsilon|^2 dx \rightharpoonup 2\sum_{i=1}^{N} \mathcal{H}^1 \lfloor_{\{re_i : r\geq 0\}}, \quad \textit{weakly as Radon measures,}$$

where e_i are as in Lemma 4.5. Moreover,

$$\sum_{i=1}^{N} e_i = 0.$$

The proof in [24] (see also [47]) can be adapted to prove this proposition. Note that the blowing down limit is unique, i.e. independent of subsequences of $\varepsilon \to 0$. In fact, by the convexity of Ω^c, the blowing down limit (in the Hausdorff distance)

$$D_i^\infty := \lim_{\varepsilon\to 0} \varepsilon D_i = \{re_i : r \geq 0\}.$$

Moreover, the blowing down limit of $(|\nabla u|^2 + 1)\chi_{D_i} dx$ is $2\mathcal{H}^1\lfloor_{\{re_i:r\geq 0\}}$.

5 Refined asymptotics

In this section we prove the exponential convergence of u to its ends (one dimensional solutions) at infinity.

Take a large R and a connected component of $\Omega \setminus B_R$, which we assume to be

$$\mathcal{C} := \{(x_1, x_2) : f_-(x_1) < x_2 < f_+(x_1),\ x_1 > R\},$$

where $f_\pm$ are convex (concave, respectively) functions defined on $[R, +\infty)$.

By Lemma 3.4,

$$\lim_{x_1\to+\infty}(f_+(x_1)-f_-(x_1))=2. \tag{5.1}$$

Then because $f'_+(x_1)$ is non-increasing in x_1 and $f'_-(x_1)$ non-decreasing in x_1, both the limits

$$\lim_{x_1\to+\infty} f'_\pm(x_1)$$

exist. Moreover, by (5.1), these two limits coincide, which can be assumed to be 0 after a rotation.

In the following we will ignore other components of $\{-1<u<1\}$, thus assume $u\equiv\pm1$ outside $\mathcal{C}$. By the regularity theory in [1] and [28], both f_+ and f_- are smooth. Then by standard elliptic estimates,

$$|\nabla^2 u(x)|\leq C, \quad \text{in } \mathcal{C}. \tag{5.2}$$

By these facts and Lemma 3.4, the limit at infinity of translations of u along $f_-(x_1)$ must be $g(x_2)$. Hence, by (5.2) and the uniform smoothness of free boundaries, we get the uniform convergence

$$\lim_{x\in\mathcal{C},|x|\to+\infty}|\nabla u|=1. \tag{5.3}$$

It should be emphasized that, in the above setting and the following proof, we do not need any kind of stability condition.

Let

$$v:=1-|\nabla u|,$$

which vanishes on $\partial\mathcal{C}$.

Direct calculation gives

$$\Delta v=-\frac{|\nabla^2 u|^2-|\nabla|\nabla u||^2}{|\nabla u|}. \tag{5.4}$$

Differentiating in x_1 twice leads to

$$\frac{1}{2}\frac{d^2}{dx_1^2}\int_{f_-(x_1)}^{f_+(x_1)} v(x_1,x_2)^2dx_2=\int_{f_-(x_1)}^{f_+(x_1)}\Big|\frac{\partial v}{\partial x_1}(x_1,x_2)\Big|^2+v(x_1,x_2)\frac{\partial^2 v}{\partial x_1^2}(x_1,x_2)dx_2.$$

Substituting (5.4) into this and integrating by parts, we get

$$\frac{1}{2}\frac{d^2}{dx_1^2}\int_{f_-(x_1)}^{f_+(x_1)} v^2=\int_{f_-(x_1)}^{f_+(x_1)}|\nabla v|^2-v\frac{|\nabla^2 u|^2-|\nabla|\nabla u||^2}{|\nabla u|}. \tag{5.5}$$

We have

$$|\nabla v|^2=\frac{|\nabla^2 u\cdot\nabla u|^2}{|\nabla u|^2}\geq|\nabla^2 u\cdot\nabla u|^2, \tag{5.6}$$

because $|\nabla u| \leq 1$. On the other hand, by denoting $\nu := \nabla u/|\nabla u|$ (recall that we can assume $|\nabla u| \geq 1/2$ in $\mathcal{C}$) and $\nu^\perp$ its rotation by angle $\pi/2$,

$$\begin{aligned} |\nabla^2 u|^2 &= |\nabla^2 u \cdot \nu|^2 + |\nabla^2 u \cdot \nu^\perp|^2 \\ &= 2\frac{|\nabla^2 u \cdot \nabla u|^2}{|\nabla u|^2} \\ &\leq 8|\nabla^2 u \cdot \nabla u|^2, \end{aligned} \tag{5.7}$$

where we have used the fact that, by the equation $\Delta u = 0$,

$$\nabla^2 u(\nu, \nu) = -\nabla^2 u(\nu^\perp, \nu^\perp), \quad \text{in } \mathcal{C}.$$

After enlarging R, we can assume $v \leq 1/64$ in $\mathcal{C}$. Combining (5.5) and (5.7) we obtain

$$|\nabla v|^2 - v\frac{|\nabla^2 u|^2 - |\nabla|\nabla u||^2}{|\nabla u|} \geq \frac{1}{2}|\nabla v|^2.$$

Hence

$$\frac{d^2}{dx_1^2}\int_{f_-(x_1)}^{f_+(x_1)} v^2 \geq \int_{f_-(x_1)}^{f_+(x_1)} |\nabla v|^2. \tag{5.8}$$

Because $v(x_1, \cdot) = 0$ on $f_-(x_1)$ and $f_+(x_1)$ and $f_+(x_1) - f_-(x_1) \leq 4$, we have the following Poincare inequality:

$$\int_{f_-(x_1)}^{f_+(x_1)} \left|\frac{\partial v}{\partial x_1}(x_1, x_2)\right|^2 dx_2 \geq \frac{\pi^2}{16}\int_{f_-(x_1)}^{f_+(x_1)} v(x_1, x_2)^2 dx_2.$$

Thus

$$\frac{d^2}{dx_1^2}\int_{f_-(x_1)}^{f_+(x_1)} v(x_1, x_2)^2 dx_2 \geq \frac{\pi^2}{16}\int_{f_-(x_1)}^{f_+(x_1)} v(x_1, x_2)^2 dx_2. \tag{5.9}$$

Because

$$\lim_{x_1 \to +\infty}\int_{f_-(x_1)}^{f_+(x_1)} v(x_1, x_2)^2 dx_2 = 0,$$

this differential inequality (5.9) implies that

$$\int_{f_-(x_1)}^{f_+(x_1)} v(x_1, x_2)^2 dx_2 \leq Ce^{-\frac{\pi}{4}x_1}, \quad \forall x_1 \text{ large}.$$

Take a nonnegative function $\eta \in C_0^\infty(-2, 2)$ with $\eta \equiv 1$ in $(-1, 1)$. For any t large, testing (5.8) with $\eta(x_1 + t)$ and integrating by parts, we obtain

$$\int_{t-1}^{t+1}\int_{f_-(x_1)}^{f_+(x_1)} |\nabla v(x_1, x_2)|^2 dx_2 dx_1 \leq Ce^{-\frac{\pi}{4}t}.$$

By (5.6) and (5.7) and using the Cauchy inequality, the above inequality implies that

$$\int_{t-1}^{t+1}\int_{f_-(x_1)}^{f_+(x_1)}|\nabla^2 u(x_1,x_2)|dx_2dx_1 \le Ce^{-\frac{\pi}{8}t}.$$

Integrating this from x_1 to $+\infty$, we obtain

$$\int_{t-1}^{t+1}\int_{f_-(x_1)}^{f_+(x_1)}\Big|\frac{\partial u}{\partial x_1}(x_1,x_2)\Big|dx_2dx_1 \le Ce^{-\frac{\pi}{8}t},$$

which can also be strengthened to

$$\int_{t}^{+\infty}\int_{-\infty}^{+\infty}\Big|\frac{\partial u}{\partial x_1}(x_1,x_2)\Big|dx_2dx_1 \le Ce^{-\frac{\pi}{8}t}.$$

In the above we have used the fact that $\frac{\partial u}{\partial x_1}=0$ outside $\mathcal{C}$.

Now the existence of the limit

$$u_\infty(x_2) := \lim_{x_1\to+\infty} u(x_1,x_2)$$

follows. Moreover,

$$\int_{-\infty}^{+\infty}\big|u(x_1,x_2)-u_\infty(x_1,x_2)\big|dx_2 \le Ce^{-\frac{\pi}{8}x_1}.$$

By the uniform Lipschitz bound on u, this can also be lifted to the convergence in $L^\infty(\mathbb{R})$.

Since $u_\infty(x_2)=g(x_1-t)$ for some constant t, by noting the nondegeneracy condition on g (i.e. $g'=1$) and a corresponding one for u, if x_1 large,

$$u(x_1,x_2)+1 \ge \frac{1}{2}\left(x_2-f_-(x_1)\right), \quad 1-u(x_1,x_2)\ge\frac{1}{2}\left(f_+(x_1)-x_2\right), \quad \text{in } \mathcal{C},$$

which follows from the fact $\frac{\partial u}{\partial x_2}\ge 1/2$ in $\mathcal{C}$ for x_1 large. This then implies

$$\lim_{x_1\to+\infty} f_\pm(x_1)=t\pm 1.$$

Moreover, the convergence rate is exponential. This finishes the proof of Theorem 1.1.

Part II
The second problem

In dimension 1, the problem (1.6) has a solution g satisfying

$$\begin{cases} g(t)\equiv 0, \quad \text{in } (-\infty,0),\\ g'(t)>0, \quad \text{in } (0,+\infty),\\ \lim\limits_{t\to+\infty} g(t)=1. \end{cases}$$

Here the convergence rate is exponential. Hence the following quantity is well defined

$$\sigma_0 := \int_0^{+\infty} \frac{1}{2} g'(t)^2 + W(g(t)) dt < +\infty.$$

Given a unit vector e and a constant $t \in \mathbb{R}$, the trivial extension $u^*(x) := g(x \cdot e - t)$, or the function

$$u^{**}(x) := g(x \cdot e - t_1) + g(-x \cdot e + t_2), \quad -\infty < t_2 \leq t_1 < +\infty,$$

are solutions of (1.6) in $\mathbb{R}^2$. Moreover, they are stable in $\mathbb{R}^2$. We still call these solutions one dimensional.

6 Uniform estimates

In this section u denotes a solution of (1.6) in $\mathbb{R}^2$. We prove a Modica type inequality and then deduce the convexity of free boundaries. In fact, most results in this section hold for solutions in $\mathbb{R}^n$, for any $n \geq 1$.

The following result is [47, Proposition 2.1].

Proposition 6.1. *$u < 1$ in Ω.*

As in Part I, to prove the Modica inequality, we first establish a gradient bound.

Lemma 6.2. *There exists a constant C such that $|\nabla u| \leq C$ in Ω.*

Proof. First standard interior gradient estimates give

$$|\nabla u(x)| \leq C \quad \text{in } \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > 1\}.$$

For x_0 near $\partial\Omega$, denote $h := \mathrm{dist}(x_0, \partial\Omega)$ and assume this distance is attained at $y_0 \in \partial\Omega$. Let

$$\tilde{u}(x) := \frac{1}{h} u(x_0 + hx).$$

Then $\tilde{u}$ is positive in $B_1(0)$, where

$$\Delta \tilde{u} = hW'(h\tilde{u}) \leq Ch.$$

Because $B_h(x_0)$ is tangent to $\partial\Omega$ at y_0,

$$1 = |\nabla \tilde{u}(z_0)| = z_0 \cdot \nabla \tilde{u}(z_0),$$

where $z_0 := (y_0 - x_0)/h$.

Applying the Hopf Lemma to $\tilde{u} - Ch|x - z_0|^2$ gives

$$1 \le C\left(\tilde{u}(0) - Ch^2\right).$$

Then by the Harnack inequality, if $h < 1/C$,

$$\sup_{B_{1/2}(0)} \tilde{u} \le C.$$

By standard interior gradient estimates,

$$|\nabla u(x_0)| = |\nabla \tilde{u}(0)| \le C.$$

□

With this gradient bound at hand, we can prove the Modica inequality.

Proposition 6.3. *In Ω,*

$$\frac{1}{2}|\nabla u|^2 \le W(u).$$

Proof. Denote $P := |\nabla u|^2/2 - W(u)$. Assume

$$\delta := \sup_{\Omega} P > 0,$$

and $x_i \in \Omega$ approaches this sup.

In Ω, P satisfies

$$\Delta P - 2\Delta u \frac{\nabla u}{|\nabla u|^2} \cdot \nabla P = |\nabla^2 u|^2 - 2\Delta u \nabla^2 u \left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right) + (\Delta u)^2 \ge 0. \tag{6.1}$$

If $\limsup \operatorname{dist}(x_i, \partial\Omega) > 0$, we can argue as in the proof of the usual Modica inequality to get a contradiction, see [35].

If $\lim \operatorname{dist}(x_i, \partial\Omega) = 0$, then $u(x_i) \to 0$. Hence for all i large,

$$\frac{1}{2}|\nabla u(x_i)|^2 \ge W(0) + \frac{\delta}{2}.$$

Then we can proceed as in the proof of Proposition 2.1 to get a contradiction. □

As in [4], the Modica inequality implies the convexity of free boundaries.

Lemma 6.4. *Each connected component of Ω^c is convex. Moreover, it is strictly convex unless u is one dimensional.*

Proof. Because $P = 0$ on $\partial\Omega$ and $P \leq 0$ in Ω,

$$P_\nu \geq 0, \quad \text{on } \partial\Omega. \tag{6.2}$$

On the other hand,

$$P_\nu = \nabla^2 u(\nabla u, \nu) - \Delta u |\nabla u| = \left(\nabla^2 u(\nu,\nu) - \Delta u\right) |\nabla u|. \tag{6.3}$$

Hence $\Delta' u := \Delta u - \nabla^2 u(\nu,\nu) \leq 0$ on $\partial\Omega$. As in [4], this implies the convexity of $\partial\Omega$.

Moreover, if $u \neq g^*$ or g^{**}, the inequality in (6.2) is strict. (This follows from an application of the Hopf lemma. Note that near $\partial\Omega$, $|\nabla u|$ has a positive lower bound, hence the second term in (6.1) is regular.) Then $\Delta' u > 0$ strictly on $\partial\Omega$, and the strict convexity of $\partial\Omega$ follows. $\square$

As in Part I, a direct consequence of this convexity is:

Corollary 6.5. Ω *is unbounded.*

Next, let

$$\Psi(x) := \begin{cases} g^{-1} \circ u(x), & x \in \Omega, \\ 0, & x \in \Omega. \end{cases}$$

By the Modica inequality, $|\nabla \Psi| \leq 1$ in Ω. It can be directly checked that Ψ satisfies

$$\Delta \Psi = f(\Psi)\left(1 - |\nabla \Psi|^2\right), \quad \text{in } \Omega,$$

where $f(\Psi) = W'(g(\Psi))/\sqrt{2W(g(\Psi))}$.

For applications below, we present a non-degeneracy result for the zero set $\{u = 0\}$.

Proposition 6.6. *If $u \neq u^{**}$ (for a unit vector e and two constants $t_1 = t_2$), then for every $x \in \partial\Omega$ and $r > 0$, $|B_r(x) \cap \{u = 0\}| > 0$.*

Proof. Assume there is an $x_0 \in \partial\Omega$ and $r_0 > 0$, such that $|B_{r_0}(x_0) \cap \{u = 0\}| = 0$. Then by the convexity of $\partial\Omega$, $\partial\Omega \cap B_{r_0}(x_0)$ are straight lines. By Lemma 6.4, in $B_{r_0}(x_0)$, $u = u^{**}$ for a unit vector e and two constants $t_1 = t_2$. Then by the unique continuation principle applied to u, this holds everywhere in $\mathbb{R}^2$. $\square$

In the following, we will always assume u satisfies the above non-degeneracy condition.

7 The stable De Giorgi conjecture

In this section we assume u to be stable outside a compact set. We use the stability condition to derive an integral curvature bound and use this to study the convergence of translations of a solution u to (1.6).

Let us first prove the stable De Giorgi conjecture, Theorem 1.6.

Proof of Theorem 1.6. As in the proof of Theorem 1.2, the stability condition implies the existence of a positive function $\varphi \in C^\infty(\overline{\Omega})$, satisfying

$$\begin{cases} \Delta\varphi = W''(u)\varphi, & \text{in } \Omega, \\ \varphi_\nu = -\left(\dfrac{W'(0)}{\sqrt{2W(0)}} - H\right)\varphi, & \text{on } \partial\Omega. \end{cases} \tag{7.1}$$

By direct differentiation, for any unit vector e, the directional derivative $u_e := e \cdot \nabla u$ also satisfies this equation.

Let $\psi := u_e/\varphi$. It still satisfies (3.1). The following proof is exactly the same as in the proof of Theorem 1.2. $\square$

The following result is similar to Lemma 3.1.

Lemma 7.1. *For any $L > 1$, there exists an $R(L)$ such that, there is no bounded component of $\{u = 0\}$ contained in $B_{R(L)}(0)^c$ with diameter smaller than L.*

Proof. **Step 1.** For any R large, take a $\varphi \in C_0^\infty(\mathbb{R}^2 \setminus B_R)$ and test the stability condition with $\varphi|\nabla u|$. After some integration by parts, we obtain

$$\begin{aligned} & -2W(0)\int_{\partial\Omega}\varphi^2\left(\frac{W'(0)}{\sqrt{2W(0)}} - H\right) \\ \leq & \int_\Omega |\nabla\varphi|^2|\nabla u|^2 + 2\varphi|\nabla u|\nabla\varphi\cdot\nabla|\nabla u| + \varphi^2|\nabla|\nabla u||^2 + W''(u)|\nabla u|^2\varphi^2 \\ = & \int_{\partial\Omega}\frac{1}{2}\varphi^2\left(|\nabla u|^2\right)_\nu + \int_\Omega |\nabla\varphi|^2|\nabla u|^2 - \varphi^2|A|^2|\nabla u|^2. \end{aligned} \tag{7.2}$$

On $\partial\Omega$,

$$\left(|\nabla u|^2\right)_\nu = 2\sqrt{2W(0)}u_{\nu\nu} = 2\sqrt{2W(0)}W'(0) + 4W(0)H.$$

Hence (7.2) can be transformed into

$$\int_\Omega |\nabla u|^2|A|^2\varphi^2 \leq \int_\Omega |\nabla u|^2|\nabla\varphi|^2.$$

Then we can use standard log cut-off test functions to show that

$$\int_{\Omega\setminus B_R} |\nabla u|^2|A|^2 \leq \frac{C}{\log R},$$

which converges to 0 as $R \to +\infty$.

Step 2. For any $\eta \in C_0^\infty(B_R(0)^c)$,

$$\begin{aligned}\int_{\partial\Omega} H\eta &= \int_{\partial\Omega} P_\nu \eta \\ &= \int_\Omega \nabla P \nabla \eta + \Delta P \eta \qquad (7.3) \\ &= \int_\Omega \left(\nabla^2 u \nabla u - \Delta u \nabla u\right) \nabla \eta + \eta \left(|\nabla^2 u|^2 - \left|\Delta u\right|^2\right).\end{aligned}$$

Claim. At x where $\nabla u(x) \neq 0$,

$$|\nabla^2 u \nabla u - \Delta u \nabla u| \leq C|\nabla u||A|, \tag{7.4}$$

and

$$\left||\nabla^2 u|^2 - \left|\Delta u\right|^2\right| \leq C\left(|\nabla u||A| + |\nabla u|^2|A|^2\right). \tag{7.5}$$

Take the coordinates at x so that

$$\frac{\nabla u(x)}{|\nabla u(x)|} = (0, 1).$$

Then

$$|\nabla u|^2|A|^2 = \left|\frac{\partial^2 u}{\partial x_1^2}\right|^2 + \left|\frac{\partial^2 u}{\partial x_1 \partial x_2}\right|^2, \tag{7.6}$$

$$\nabla^2 u \nabla u - \Delta u \nabla u = |\nabla u|\left(\frac{\partial^2 u}{\partial x_1 \partial x_2}, -\frac{\partial^2 u}{\partial x_1^2}\right), \tag{7.7}$$

Because $|\nabla u| \leq \sqrt{2W(u)} \leq C$, (7.4) follows. Next

$$\begin{aligned}|\nabla^2 u|^2 - \left|\Delta u\right|^2 &= 2\left(\left|\frac{\partial^2 u}{\partial x_1 \partial x_2}\right|^2 - \frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 u}{\partial x_2^2}\right) \qquad (7.8) \\ &= 2\left(\left|\frac{\partial^2 u}{\partial x_1 \partial x_2}\right|^2 + \left|\frac{\partial^2 u}{\partial x_1^2}\right|^2\right) - 2W'(u)\frac{\partial^2 u}{\partial x_1^2}. \qquad (7.9)\end{aligned}$$

Because $|W'(u)| \leq C$, (7.5) follows. This finishes the proof of this **Claim.**

Assume there is a connected component of $\{u = 0\}$, D, contained in $B_R(0)^c$ with its diameter smaller than L. Take a point x in this component and η to be a standard cut-off

function in $B_{2L}(x)$ with $\eta \equiv 1$ in $B_L(x)$. Substituting this into (7.3), using (7.4) and (7.5), and noting that $H \geq 0$ on $\partial\Omega$, we obtain

$$\begin{aligned}\int_{\partial D} H &\leq C \int_{B_{2L(x)}} |\nabla u||A| + |\nabla u|^2|A|^2 \\ &\leq C \int_{B_{2L(x)}} |\nabla u|^2|A|^2 + CL \left(\int_{B_{2L(x)}} |\nabla u|^2|A|^2\right)^{\frac{1}{2}} \\ &\leq \frac{C(1+L)}{\sqrt{\log R}}.\end{aligned}$$

On the other hand, because D is convex, the Gauss-Bonnet theorem says

$$\int_{\partial D} H = 2\pi.$$

Thus we get a contradiction if

$$\frac{C(1+L)}{\sqrt{\log R}} < 2\pi. \qquad \square$$

As in Part I, we have two corollaries.

Corollary 7.2. *For any $x \in \partial\Omega \setminus B_{R(L)}(0)$ and $r \in (0, L/2)$, the connected component of $\partial\Omega \cap B_r(x)$ passing through x, denoted by $\Gamma^{x,r}$, has its boundary in $\partial B_r(x)$. Hence,*

$$\mathcal{H}^1(\Gamma^{x,r}) \geq 2r.$$

Corollary 7.3. *For any $\varepsilon > 0$ small and $L > 0$ large, there exists an $R(L,\varepsilon)$ so that the following holds. For any $x \in \partial\Omega \setminus B_{R(L,\varepsilon)}(0)$, the connected component of $\partial\Omega$ passing through x, $\Gamma^{x,L}$, satisfies*

$$\int_{\Gamma^{x,L} \cap B_L(x)} H \leq \varepsilon,$$

and

$$dist_H(\Gamma^{x,L} \cap B_L(x), \{e^{x,L} \cdot (y - x) = 0\} \cap B_L(x)) \leq \varepsilon,$$

where $e^{x,L}$ is a unit vector.

Lemma 7.4. *For any $\ell \geq 1$ and $x_k \in \partial\Omega$, $|x_k| \to \infty$, the translation function*

$$u_k(x) := u(x_k + x)$$

converges in the C^ℓ sense to $u^*(e \cdot x)$ for some unit vector e, or $u^{**}(e, 0, t)$ for a unit vector e and a constant $t \leq 0$. Moreover, the translations of Ω, $\Omega_k := \Omega - x_k$ converges to $\Omega(u^*)$ or $\Omega(u^{**})$ in the C^ℓ sense, on any compact set of $\mathbb{R}^2$.

Finally, for application in the proof of Theorem 9.1, we present a cheap Harnack inequality for the curvature of $\partial\Omega$. This is basically the consequence of boundary gradient estimates and the Hopf lemma.

Proposition 7.5. *There exist two constants R_H and C_H such that, for any $x \in \partial\Omega\backslash B_{R_H}(0)$,*

$$\sup_{B_1(x)\cap\partial\Omega} H \le C_H \inf_{B_1(x)\cap\partial\Omega} H.$$

Proof. By a direct calculation we get

$$\operatorname{div}\left(|\nabla u|^{-2}\nabla P\right) = 0, \quad \text{in } \Omega. \tag{7.10}$$

Note that P vanishes continuously on $\partial\Omega$.

By taking a sufficiently large R_H, Lemma 7.4 gives two constants $0 < \lambda < \Lambda < +\infty$ such that, for any $x \in \partial\Omega \setminus B_{R_H}(0)$, in $B_4(x) \cap \Omega$ we have

$$\lambda \le |\nabla u|^{-2} \le \Lambda, \quad |\nabla|\nabla u|^{-2}| + |\nabla^2|\nabla u|^{-2}| \le \Lambda.$$

Moreover, $\partial\Omega \cap B_4(x)$ is also uniformly bounded in C^2.

Then by standard boundary gradient estimates for elliptic equations,

$$\sup_{B_1(x)\cap\partial\Omega} P_\nu \le C \sup_{B_2(x)\cap\Omega} \left(-P\right). \tag{7.11}$$

By the Harnack inequality and boundary Harnack inequality (see [5, Theorem 11.5]),

$$\sup_{B_2(x)\cap\Omega} \left(-P\right) \le C \inf_{B_3(x)\cap\Omega_1} \left(-P\right), \tag{7.12}$$

where $\Omega_1 := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \ge 1\}$. Next, because $\partial\Omega$ is convex, for any $y \in \partial\Omega \cap B_1(x)$, there exists a ball $B_1(z) \subset \Omega \cap B_3(x)$ touching $\partial\Omega$ at y. By the Hopf lemma,

$$P_\nu(y) \ge c\left(-P(z)\right). \tag{7.13}$$

Since $z \in \Omega_1 \cap B_3(x)$, combining (7.13) with (7.12) we get

$$\inf_{\partial\Omega\cap B_1(x)} P_\nu \ge c \sup_{B_3(x)\cap\Omega} \left(-P\right). \tag{7.14}$$

Substituting (7.11) into this gives

$$\sup_{B_1(x)\cap\partial\Omega} P_\nu \le C \inf_{\partial\Omega\cap B_1(x)} P_\nu. \tag{7.15}$$

Since (6.3) implies that

$$P_\nu = 2W(0)H, \quad \text{on } \partial\Omega,$$

the conclusion of this proposition follows from (7.15). $\square$

8 Unbounded components of Ω^c are finite

In this section we establish the finiteness of unbounded connected components of Ω^c, for a solution of (1.6), which is stable outside a compact set.

By the stability of u outside a compact set (say $B_{R_0}(0)$ for some $R_0 > 0$), there exists a positive function $\varphi \in C^\infty(\overline{\Omega})$ satisfying

$$\begin{cases} \Delta\varphi = W''(u)\varphi, & \text{in } \Omega \setminus B_{R_0}(0), \\ \varphi_\nu = -\left(\dfrac{W'(0)}{\sqrt{2W(0)}} - H\right)\varphi, & \text{on } \partial\Omega \setminus B_{R_0}(0). \end{cases} \tag{8.1}$$

The following lemma is the main tool in this section.

Lemma 8.1. *Take a unit vector e and let D be a connected component of $\{u_e \neq 0\}$. Then D intersects B_{R_0}.*

Proof. Assume D dose not intersect B_{R_0}. Let ψ be the restriction of u_e to D, extended to be 0 outside Ω. Then $0 \leq \psi \leq C$ is a continuous subsolution to (7.1), because u_e is a solution to (7.1).

Then $\phi := \psi/\varphi_0$ is well defined. It is nonnegative, continuous. Moreover, it satisfies

$$\begin{cases} \operatorname{div}\left(\varphi_0^2 \nabla\phi\right) \geq 0, & \text{in } \Omega, \\ \phi_\nu \geq 0, & \text{on } \partial\Omega. \end{cases} \tag{8.2}$$

Note that the support of ϕ is contained in $B_{R_0}(0)^c$.

The same proof of Theorem 1.6, using the standard log cut off functions, gives $\phi \equiv 0$. Hence $u_e \equiv 0$ in D and we get a contradiction. $\square$

We use this lemma to prove

Lemma 8.2. *There are only finitely many unbounded connected components of Ω^c.*

Proof. Let D_α be all of the unbounded connected components of Ω^c. By our preliminary analysis, each D_α is a convex open domain with smooth boundary. Moreover, ∂D_α is a single simple curve (see the proof of Lemma 4.1). Thus we can take a point $x_\alpha \in \partial D_\alpha$ and a unit vector e_α, such that D_α can be represented by

$$D_\alpha = \{x : x \cdot e_\alpha > f(x \cdot e_\alpha^\perp - x_\alpha \cdot e_\alpha^\perp) + x_\alpha \cdot e_\alpha\},$$

where f is a convex function defined on an (connected) interval I_α containing 0. The ray

$$L_\alpha := \{x_\alpha + te_\alpha, \ t \geq 0\}$$

is contained in D_α. Hence for different α, L_α are disjoint from each other.

Take an N large and two e_{α_1}, e_{α_2}. We will prove that if

$$e_{\alpha_1} \cdot e_{\alpha_2} \geq 1 - \frac{1}{N},$$

then in the cone

$$\{x : x \cdot \frac{e_{\alpha_1} + e_{\alpha_2}}{2} \geq e_{\alpha_1} \cdot \frac{e_{\alpha_1} + e_{\alpha_2}}{2}\},$$

there are only finitely many unbounded connected components of Ω^c. It is clear that the conclusion of this lemma follows from this claim.

Now fix such a cone, which can be assumed to be $\{0 < x_2 < \varepsilon x_1\}$, with $\varepsilon \leq 1/N$. Take an R_3 large so that both $\{u_{x_2} = 0\} \cap \Omega$ and $\partial\Omega$ intersect ∂B_{R_3} transversally in this cone. (Here $u_{x_2} := \frac{\partial u}{\partial x_2}$.)

In the following we denote $\mathcal{C} := \{0 < x_2 < \varepsilon x_1\} \setminus B_{R_3}$.

The number of connected components of $\partial B_{R_3} \cap \{0 < x_2 < \varepsilon x_1\} \cap \{u_{x_2} \neq 0\} \cap \Omega$ is denoted by J. By Lemma 8.1, there are exactly J connected components of $\{u_{x_2} \neq 0\} \cap \mathcal{C}$.

Now assume the number of unbounded connected components of $\mathcal{C} \setminus \Omega$ to be larger than $J + 1$. Take $J + 1$ such components $U_0, \cdots, U_{J+1}$, where U_0 is the one containing $\{x_2 = 0, x_1 \geq R_3\}$ and U_{J+1} the one containing $\{x_2 = \varepsilon x_1, x_1 \geq R_3\}$.

For each $i \neq 0, J+1$, there exists an R_i^* such that

$$\partial U_i = \{(x_1, x_2) : x_2 = f_i^\pm(x_1), \quad \text{on } x_1 \geq R_i^*\}.$$

Here f_i^+ is convex and f_i^- concave.

For each i, perhaps after enlarging R_i^*, we can assume

$$\Big|\frac{df_i^\pm}{dx_1}(x_1)\Big| \leq 2\varepsilon, .$$

Since $u = 0$ and $|\nabla u| = \sqrt{2W(0)}$ on the curve $\{x_2 = f_i^\pm(x_1)\}$, we see

$$u_{x_2}(x_1, f_i^+(x_1)) \geq \sqrt{\frac{W(0)}{2}} > 0, \quad u_{x_2}(x_1, f_i^-(x_1)) \leq -\sqrt{\frac{W(0)}{2}} < 0.$$

The sign follows from the fact that in a neighborhood of $\{x_2 = f_i^\pm(x_1)\}$, $u > 0$ on one side and $u = 0$ on the other side. Thus for each i, there exists an R_i^{**} such that a neighborhood of $\{(x_1, x_2) : x_2 = f_i^\pm(x_1), \ x_1 \geq R_i^{**}\}$ in Ω, which we denote by $U_i^\pm$, is contained in $\{u_{x_2} > 0\}$ and $\{u_{x_2} < 0\}$ respectively.

Because the number of connected components of $\{u_{x_2} \neq 0\}$ is not larger than J, there exist $i > j$ such that, U_i^+ and U_j^+ are contained in the same connected component of $\{u_{x_2} \neq 0\}$. By the connectedness and the unboundedness of U_i^+ and U_j^+, there exists a smooth embedded curve contained in the component containing U_i^+ and U_j^+. This curve separates U_i^- from B_{R_3}. This contradicts Lemma 8.1 and the claim is proved. $\square$

Corollary 8.3. *There exists a constant C such that, for any $R > 0$ large, the number of connected components of $\Omega \setminus B_R$ is smaller than C.*

Proof. First, because $\partial\Omega$ is strictly convex with respect to the outward normal vector, each component of $\Omega \setminus B_R$ is unbounded. Hence it has an unbounded component of $\partial\Omega$ as its boundary.

On the other hand, the previous lemma says that each unbounded component of $\Omega^c \setminus B_R$ have at most two unbounded bounded components of $\partial\Omega \setminus B_R$.

Thus the number of connected components of $\Omega \setminus B_R$ is at most two times the number of unbounded components of Ω^c. $\square$

In the following, unbounded connected components of Ω^c are denoted by D_i, $1 \leq i \leq K$ for some K. Each D_i contains a ray $L_i = \{x : x = x_i^* + re_i, r \geq 0\}$, where $x_i^* \in \partial D_i$ and e_i is a unit vector.

Lemma 8.4. *If $i \neq j$, $e_i \neq e_j$.*

Proof. Assume by the contrary, there are two different components of Ω^c, D_1 and D_2, such that

$$\{(x_1, x_2) : x_2 = t_i, x_1 \geq R\} \subset D_i, \quad i = 1, 2,$$

where $t_1 < t_2$.

There is a part of ∂D_i having the form

$$x_2 = f_i(x_1), \quad x_1 \geq R,$$

where $t_2 > f_2(x_1) > f_1(x_1) > t_1$. Here f_1 is concave and f_2 convex. Hence f_1 is eventually increasing in x_1 and f_2 eventually decreasing in x_1. Thus their limits as $x_1 \to +\infty$ exist. Moreover,

$$t_2 > \lim_{x_1 \to +\infty} f_2(x_1) \geq \lim_{x_1 \to +\infty} f_1(x_1) > t_1. \tag{8.3}$$

By the above choice, $u > 0$ in a neighborhood below $\{x_2 = f_2(x_1)\}$ and above $\{x_2 = f_1(x_1)\}$.

By Proposition 7.4, as $t \to +\infty$,

$$u^t(x_1, x_2); = u(x_1 + t, x_2 + f_1(t))$$

converges to a one dimensional solution. In particular, for any $L > 0$, if x_1 is large enough, $u > 0$ in $\{(x_1, x_2) : f_1(x_1) < x_2 < f_1(x_1) + L\}$. However this contradicts (8.3) and the proof is completed. $\square$

Lemma 8.5. *Let $\mathcal{C} = \{(x_1, x_2) : |x_2| < \lambda x_1, x_1 > R\}$ for some $\lambda > 0$ and $R > 0$. Assume that $u = 0$ in $\mathcal{C} \cap \{x_2 > f_+(x_1)\}$ and $\mathcal{C} \cap \{x_2 < f_-(x_1)\}$, where $f_\pm$ are convex (concave, respectively) functions defined on $[R, +\infty)$, satisfying*

$$-\lambda x_1 < f_-(x) < f_+(x) < \lambda x_1.$$

Then both the limits $\lim_{x_1 \to +\infty} f'_\pm(x_1)$ exist. Moreover,

$$-\lambda \leq \lim_{x_1 \to +\infty} f'_-(x_1) < \lim_{x_1 \to +\infty} f'_+(x_1) \leq \lambda.$$

Proof. Since f_+ is convex, $f'_+(x_1)$ is increasing in x_1. Because $f_+(x_1) \leq \lambda x_1$, it is easy to see that $f'_+(x_1) \leq \lambda$ for all $x_1 > R$, by using the convexity of f_+. The existence of $\lim_{x_1 \to +\infty} f'_+(x_1)$ then follows. For f_- we have similar statements.

Next assume

$$\lim_{x_1 \to +\infty} f'_-(x_1) = \lim_{x_1 \to +\infty} f'_+(x_1). \tag{8.4}$$

Then by noting that $f_+ - f_-$ is positive and convex, for all $x_1 > R$,

$$f'_+(x_1) - f'_-(x_1) \leq \lim_{x_1 \to +\infty} \big(f'_+(x_1) - f'_-(x_1)\big) = 0.$$

Hence $\lim_{x_1 \to +\infty} (f_+(x_1) - f_-(x_1))$ exists and this limit lies in $[0, f_+(0) - f_-(0)]$. However, by Proposition 7.4, we know that as $x_1 \to +\infty$, $u(x_1 + y_1, f_-(x_1) + y_2)$ converges to u^* or u^{**} uniformly on any compact set of $\mathbb{R}^2$. In particular, we should have

$$\lim_{x_1 \to +\infty} \big(f'_+(x_1) - f'_-(x_1)\big) = +\infty.$$

This is a contradiction. Thus the assumption (8.4) does not hold. $\square$

9 Bounded components of Ω^c are finite

In this section we establish a decay estimate for the curvature $H\lfloor_{\partial\Omega}$. Then we use this to prove the finiteness of bounded components of Ω^c.

Theorem 9.1. *Let u be a solution of (1.6) which is stable outside a compact set. There exists a constant C such that*

$$H(x) \leq \frac{C}{1 + |x|}, \qquad \forall\, x \in \partial\Omega.$$

The proof of this theorem is quite involved and we postpone it to Part III. Roughly speaking, we first use the doubling lemma of Polácick-Quittner-Souplet [37] to reduce the proof to the following setting:

1. u_ε is a solution of
$$\begin{cases} \varepsilon \Delta u_\varepsilon = \dfrac{1}{\varepsilon} W'(u_\varepsilon), & \text{in } \{u_\varepsilon > 0\} \cap Q_1(0), \\ |\nabla u_\varepsilon| = \dfrac{1}{\varepsilon}\sqrt{2W(0)}, & \text{on } \partial\{u_\varepsilon > 0\} \cap Q_1(0). \end{cases}$$
Here $Q_1(0) = \{|x_1| < 1, |x_2| < 1\}$.
2. There exists a constant $C > 0$ such that
$$\int_{Q_1(0)} \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon)\chi_{\{u_\varepsilon>0\}} \le C. \tag{9.1}$$
3. u_ε satisfies the Modica inequality,
$$\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 \le \frac{1}{\varepsilon} W(u_\varepsilon), \quad \text{in } \{u_\varepsilon > 0\}.$$
Hence $\partial\{u_\varepsilon > 0\}$ is convex.
4. The curvature of $\partial\{u_\varepsilon > 0\}$ is bounded by 4, and it equals 1 at the origin.

The last condition says the free boundaries are uniformly bounded in $C^{1,1}$. Because free boundaries converge to lines in low regularity spaces, by some interior regularity results we show that the curvature at the origin converges to 0, hence a contradiction is obtained. Note that the last claim would follow directly if we have a kind of uniform $C^{2,\alpha}$ regularity for $\partial\{u_\varepsilon > 0\}$.

By noting that in dimension 2, minimal surfaces are exactly straight lines, which have zero curvature, we can improve the conclusion of the previous theorem to

Corollary 9.2. *Under the assumptions in the previous theorem,*
$$H(x) = o\left(\frac{1}{1+|x|}\right), \qquad \text{as } \ x \in \partial\Omega \text{ and } |x| \to +\infty.$$

Proof. Assume there exists $x_k \in \partial\Omega$, $|x_k| \to +\infty$ such that
$$\lim_{k\to+\infty} |x_k| H(x_k) > 0.$$
Define
$$u_k(x) := u\left(x_k + \frac{x}{|x_k|}\right).$$
By Theorem 9.1, the curvature of $\partial\{u_k > 0\} \cap B_{1/2}(0)$ is uniformly bounded, while the curvature at 0 converges to a positive constant. The remaining proof is exactly the same as in the proof of Theorem 9.1. ∎

Using this corollary we prove

Proposition 9.3. *There exists an $R_4 > 0$ such that all bounded connected components of Ω^c are contained in $B_{R_4}(0)$.*

Proof. Assume by the contrary, there are infinitely many bounded connected components of Ω^c, denoted by $\mathcal{G}_k$, such that

$$\max_{x \in \overline{\mathcal{G}_k}} |x| \to +\infty.$$

Let $x_k \in \overline{\mathcal{G}_k}$ attain this maxima. By the previous corollary,

$$H(x_k) = o\left(\frac{1}{|x_k|}\right).$$

On the other hand, since $\mathcal{G}_k$ is contained in $B_{|x_k|}(0)$ and these two sets touch at x_k,

$$H(x_k) \geq H\lfloor_{\partial B_{|x_k|}(0)}(x_k) = \frac{1}{|x_k|}.$$

This is a contradiction. $\square$

10 Blowing down analysis

In this section we perform the blowing down analysis and gives a description of the blowing down limit.

In the previous two sections we have established the finiteness of unbounded and bounded connected components of Ω^c. Now we prove the natural energy growth bound.

Theorem 10.1. *There exists a constant C such that, for any $R > 1$,*

$$\int_{B_R} \frac{1}{2}|\nabla u|^2 + W(u)\chi_\Omega \leq CR. \tag{10.1}$$

Proof. By Proposition 9.3, all bounded connected components of Ω are contained in $B_{R_4}(0)$. There are two cases, depending wether there are unbounded components of Ω^c.

Case 1. First assume there is no unbounded component of Ω^c, hence $u > 0$ in $B_{R_4}(0)^c$. For any $x_k \to +\infty$, let

$$u_k(x) := u(x_k + x).$$

Assume it (up to a subsequence) converges to a limit u_∞ in $C_{loc}(\mathbb{R}^2)$. Because for any $R > 0$, if k large, $u_k > 0$ in $B_R(0)$, we have

$$\Delta u_k = W'(u_k) \quad \text{in } B_R(0).$$

By standard elliptic regularity and Arzela-Ascoli theorem, u_k converges to u_∞ in $C^2_{loc}(\mathbb{R}^2)$. Hence, by noting that R can be arbitrarily large, we get

$$\Delta u_\infty = W'(u_\infty) \quad \text{in } \mathbb{R}^2.$$

As in Lemma 7.4, u_∞ is stable. Then similar to Theorem 1.6, u_∞ is one dimensional. After a rotation, assume it to be a function of x_1 only. Hence u_∞ satisfies

$$\frac{d^2 u_\infty}{dx_1^2} = W'(u_\infty) \quad \text{on } \mathbb{R}. \tag{10.2}$$

By noting that $u_\infty \geq 0$, it is easily seen that $u_\infty \equiv 1$.

Since this limit is independent of $x_k \to \infty$, we obtain the uniform convergence

$$\lim_{|x|\to+\infty} u(x) = 1.$$

In particular, there exists an $\tilde{R}_E > 0$ such that $u > \gamma$ outside $B_{\tilde{R}_E}(0)$. Direct calculation gives

$$\Delta\left(1-\tilde{u}\right) \geq c\left(1-\tilde{u}\right) \quad \text{outside } B_{\tilde{R}_E}(0).$$

From this differential inequality we deduce that

$$1 - u(x) \leq Ce^{-c|x|} \quad \text{outside } B_{\tilde{R}_E}(0). \tag{10.3}$$

By standard elliptic regularity, we also have

$$|\nabla u(x)| \leq Ce^{-c|x|} \quad \text{outside } B_{\tilde{R}_E}(0). \tag{10.4}$$

Because u is a classical solution, we have the following Pohozaev identity:

$$2\int_{B_R(0)} W(u)\chi_{\{u>0\}} = R\int_{\partial B_R(0)} |\nabla u|^2 - 2\Big|\frac{x}{|x|}\cdot\nabla u(x)\Big|^2 + 2W(u)\chi_{\{u>0\}}, \quad \forall R>0. \tag{10.5}$$

Letting $R \to +\infty$, by (10.3) and (10.4), the right hand side converges to 0 exponentially. This leads to

$$\int_{\mathbb{R}^2} W(u)\chi_{\{u>0\}} = 0.$$

This is only possible if $u \equiv 1$ or $u \equiv 0$.

Case 2. Now assume there are unbounded components of Ω^c. Take an unbounded component of $\Omega \setminus B_{R_4}(0)$, denoted by $\mathcal{C}$. Its boundary consists of a part of $\partial B_{R_4}(0)$ and two convex curves, denoted by $\Gamma^\pm$. Assume $\Gamma^\pm$ lies on the boundary of $D^\pm$, two connected components of Ω^c. (We do not claim these two components of Ω^c to be different.) Let $L^\pm$

be two rays, contained strictly in $D^{\pm}$. Assume L^- to be $\{(x_1,x_2):x_2=0, x_1>R_E\}$ for some $R_E>R_4$. Then Γ^- has the form

$$\{(x_1,x_2):x_2=f_-(x_1)\},$$

where $f_-\geq 0$ is a concave function defined on $(R_E,+\infty)$.

By the concavity, $\lim_{x_1\to+\infty} f_-'(x_1)$ exists, which is nonnegative because $f_-\geq 0$.

Subcase 2.1 If the angle between L_+ and L_- is smaller than $\pi/2$, then Γ^+ can also be represented by

$$\{(x_1,x_2):x_2=f_+(x_1)\},$$

where $f_+>f_-$ is a convex function defined on $(R_E,+\infty)$.

By Lemma 8.5,

$$\lim_{x_1\to+\infty} f_+'(x_1)>\lim_{x_1\to+\infty} f_-'(x_1).$$

Hence, by letting

$$\lambda:=\frac{1}{2}\left(\lim_{x_1\to+\infty} f_-'(x_1)+\lim_{x_1\to+\infty} f_+'(x_1)\right),$$

there exist two constants $\bar{R}_E\geq R_4$ and $t\in\mathbb{R}$ so that the ray

$$L^*:=\{(x_1,x_2):x_2=\lambda x_1+t, x_1>\bar{R}_E\}$$

belongs to $\mathcal{C}$.

Because $u>0$ in $\mathcal{C}$, similar to the derivation of (10.3), we have

$$1-u(x)\leq Ce^{-c(x_1-\bar{R}_E)}\quad \text{on } L^*. \tag{10.6}$$

We claim that there exists a constant C such that

$$H(x_1):=\int_0^{\lambda x_1+t}\frac{u_{x_2}^2-u_{x_1}^2}{2}+W(u)\chi_\Omega dx_2\equiv C+O(e^{-c(x_1-\bar{R}_E)}),\quad \forall\ x_1>\bar{R}_E. \tag{10.7}$$

This is the Hamiltonian identity and it can be proved by differentiation and integration by parts. The detailed calculation is postponed to the proof of Proposition 11.4 below.

By the Modica inequality, (10.7) implies that

$$\int_0^{\lambda x_1+t}\Big|\frac{\partial u}{\partial x_2}(x_1,x_2)\Big|^2 dx_2\leq C,\quad \forall\ x_1>\bar{R}_E. \tag{10.8}$$

By rotating the plane a little, we also get for a small δ (depending on λ), such that

$$\int_0^{\lambda x_1+t}\Big|\frac{\partial u}{\partial x_2}(x_1,x_2)\Big|^2+\delta\Big|\frac{\partial u}{\partial x_1}(x_1,x_2)\Big|^2 dx_2\leq 2C,\quad \forall\ x_1>\bar{R}_E. \tag{10.9}$$

Combining these two we see, for any $R > \bar{R}_E$,

$$\int_{B_R \cap \{f_+(x_1) < x_2 < \lambda x_1 + t\}} |\nabla u|^2 \leq CR. \tag{10.10}$$

Because

$$\int_{B_R \cap \{f_+(x_1) < x_2 < \lambda x_1 + t\}} \frac{u_{x_2}^2 - u_{x_1}^2}{2} + W(u)\chi_\Omega \leq CR, \quad \forall R > \bar{R}_E,$$

adding (10.10) into this we obtain

$$\int_{B_R \cap \{f_+(x_1) < x_2 < \lambda x_1 + t\}} \frac{|\nabla u|^2}{2} + W(u)\chi_\Omega \leq CR, \quad \forall R > \bar{R}_E.$$

A similar one holds for the energy in $B_R \cap \{\lambda x_1 + t < x_2 < f_-(x_1)\}$. This shows that the energy in $B_R \cap \mathcal{C}$ grows linearly in R.

Subcase 2.2 Assume the angle between $L^\pm$ is not smaller than $\pi/2$. In this case we can take two different rays $\tilde{L}^\pm$, totally contained in $\mathcal{C}$, so that the angle between L^+ and $\tilde{L}^+$ (and L^- between $\tilde{L}^-$) is smaller than $\pi/2$. These two rays dividing $\mathcal{C}$ into three subdomains:

1. $\mathcal{C}^+$, bounded by Γ^+ and $\tilde{L}^+$;
2. $\mathcal{C}^0$, bounded by $\tilde{L}^+$ and $\tilde{L}^-$;
3. $\mathcal{C}^-$, bounded by $\tilde{L}^-$ and Γ^-.

The energy in $\mathcal{C}^\pm$ can be estimated as in Subcase 2.1. In $\mathcal{C}^0$, similar to (10.3), we have

$$W(u(x)) \leq C\,(1 - u(x))^2 \leq Ce^{-c|x|}.$$

Hence

$$\int_{\mathcal{C}^0} W(u) < +\infty.$$

By the Modica inequality,

$$\int_{\mathcal{C}^0} \frac{1}{2}|\nabla u|^2 + W(u) \leq 2\int_{\mathcal{C}^0} W(u) < +\infty.$$

Combining this with the energy estimate in $\mathcal{C}^\pm$ we get the linear growth energy bound in $\mathcal{C} \cap B_R$, for any $R > 1$.

Since there are only finitely many unbounded components of Ω^c, putting Subcase 2.1 and Subcase 2.2 together we get the linear energy growth bound. $\square$

Some remarks are in order.

Remark 10.2. *If u is nontrivial, there must exist unbounded connected components of Ω^c.*

In the following we show that unless u is one dimensional, for all R large, there are at least two unbounded components of $\Omega^c \setminus B_R$.

For each $\varepsilon > 0$, let

$$u_\varepsilon(x) = u(\varepsilon^{-1}x),$$

and $\Omega_\varepsilon := \varepsilon\Omega$.

By Theorem 10.1, as $\varepsilon \to 0$, the measures

$$\mu_\varepsilon := \left(\frac{\varepsilon|\nabla u_\varepsilon|^2}{2} + W(u_\varepsilon)\chi_{\{u_\varepsilon>0\}}\right) dx$$

have uniformly bounded mass on any compact of $\mathbb{R}^2$. Hence we can assume that, perhaps after passing to a subsequence, it converges to a positive Radon measure μ, weakly on any compact set of $\mathbb{R}^2$.

For application below, we also assume that as Radon measures,

$$\varepsilon|\nabla u_\varepsilon|^2 dx \rightharpoonup \mu_1,$$

$$\frac{1}{\varepsilon}W(u_\varepsilon)dx \rightharpoonup \mu_2,$$

on any compact set of $\mathbb{R}^2$. Note that $\mu = \mu_1/2 + \mu_2$. In the following we denote $\Sigma = \mathrm{spt}\mu$.

Furthermore, we can also assume the matrix valued measures

$$\varepsilon\nabla u_\varepsilon \otimes \nabla u_\varepsilon dx \rightharpoonup [\tau_{\alpha\beta}]\mu_1,$$

where $[\tau_{\alpha\beta}]$, $1 \le \alpha, \beta \le 2$, is measurable with respect to μ_1. Moreover, τ is nonnegative definite μ_1-almost everywhere and

$$\sum_{\alpha=1}^{2} \tau_{\alpha\alpha} = 1, \quad \mu_1 - a.e.$$

By the Hutchinson-Tonegawa theory (see [47]), Σ is countably 1-rectifiable and $I - \tau = T_x\Sigma$ $\mathcal{H}^1$-a.e. on Σ.

Define the varifold V by

$$< V, \Phi >:= \int_\Sigma \Phi(x, T_x\Sigma)\Theta(x)d\mathcal{H}^1.$$

Then V is stationary (see [47] again).

Moreover, μ can be represented by

$$\mu = \Theta \mathcal{H}^1 \lfloor_{\Sigma},$$

where Θ/σ_0 are positive integers $\mathcal{H}^1$-a.e. on Σ.

In the above Θ is defined by

$$\Theta(x) := \lim_{r\to 0} \frac{\mu(B_r(x))}{r}.$$

The existence of this limit is guaranteed by the monotonicity of $\mu(B_r(x))/r$ (the monotonicity formula for stationary varifolds).

In our setting, because u_ε is the blowing down sequence constructed from u, for any $r > 0$,

$$\frac{\mu(B_r(0))}{r} = \lim_{\varepsilon\to 0} \frac{\mu_\varepsilon(B_r(0))}{r} = \lim_{r\to+\infty} \frac{\int_{B_r(0)} \frac{1}{2}|\nabla u|^2 + W(u)\chi_\Omega}{r},$$

which is a constant independent of r, thanks to the monotonicity formula for u (see for example [47, Proposition 2.4]) and the energy growth bound Theorem 10.1. Then by the monotonicity formula for stationary varifolds, we can show that Σ is a cone with respect to the origin. Hence we have the following characterization of the blowing down limit.

Proposition 10.3. *There exist a finite number of unit vectors e_α^{**} and positive integers n_α, such that*

$$\mu = \sigma_0 \sum_\alpha n_\alpha \mathcal{H}^1 \lfloor_{\{re_\alpha^{**}, r\geq 0\}}.$$

Moreover, we have the balancing formula

$$\sum_\alpha n_\alpha e_\alpha^{**} = 0.$$

The balancing formula is equivalent to the stationary condition for the varifold V.

In the following we assume e_α^{**} are in clockwise order.

Remark 10.4. *If there are only two unit vectors e_1^* and e_2^*, then*

$$e_1^* = -e_2^*, \quad \textit{and} \quad n_1 = n_2.$$

Let

$$w(x) := \Phi(u(x)) = \int_0^{u(x)} \sqrt{2W(t)}dt,$$

and $w_\varepsilon(x) := w(\varepsilon^{-1}x)$. For any $R > 0$,

$$\int_{B_R(0)} |\nabla w_\varepsilon| \quad = \quad \int_{B_R(0)} \sqrt{2W(u_\varepsilon)} |\nabla u_\varepsilon|$$

$$\leq \int_{B_R(0)} \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\chi_{\{u_\varepsilon>0\}} \leq CR.$$

Since $0 \leq w_\varepsilon \leq \int_0^1 \sqrt{2W(t)}dt$, it is uniformly bounded in $BV_{loc}(\mathbb{R}^2)$. Then up to a subsequence w_ε converges in $L^1_{loc}(\mathbb{R}^n)$ to a function $w_\infty \in BV_{loc}(\mathbb{R}^n)$.

By extending Φ suitably to (-1,1), there exists a continuous inverse of it. Then $u_\varepsilon = \Phi^{-1}(w_\varepsilon)$ converges to $\Phi^{-1}(w_\infty)$ in $L^1_{loc}(\mathbb{R}^2)$. Since

$$\int_{B_1} W(u_\varepsilon)\chi_{\{u_\varepsilon>0\}} \leq C\varepsilon,$$

$u_\varepsilon \to 0$ or 1 a.e. in B_1. Hence there exists a measurable set Ω_∞ such that

$$u_\varepsilon \to \chi_{\Omega_\infty}, \quad \text{in } L^1_{loc}(\mathbb{R}^2).$$

Because $w_\infty = (\int_0^1 \sqrt{2W(t)}dt)\chi_\Omega$, $\chi_\Omega \in BV_{loc}(\mathbb{R}^n)$.

As $\varepsilon \to 0$, $\Psi_\varepsilon(x) := \varepsilon\Psi(\varepsilon^{-1}x)$ converges uniformly to a function Ψ_∞ on any compact set of $\mathbb{R}^2$. By the *vanishing viscosity method*, we can prove that, in the open set $\{\Psi_\infty > 0\}$, Ψ_∞ is a viscosity solution of the eikonal equation

$$|\nabla\Psi_\infty|^2 - 1 = 0.$$

See [46, Appendix A] for more details.

Lemma 10.5. $\Psi_\infty = 0$ *on* Σ.

Proof. Assume by the contrary, $\Psi_\infty(x_0) > 0$ for some $x_0 \in \Sigma$. Then there exists a ball $B_r(x_0)$ such that Ψ_∞ has a positive lower bound in this ball. By the uniform convergence of Ψ_{ε_k}, for all ε_k small, Ψ_{ε_k} also has a uniform positive lower bound in this ball. Then by the definition of Ψ_{ε_k},

$$u_{\varepsilon_k} \geq 1 - Ce^{-c\varepsilon_k^{-1}} \quad \text{in this ball.}$$

From this it can be checked directly that

$$\frac{\varepsilon_k}{2}|\nabla u_{\varepsilon_k}|^2 + \frac{1}{\varepsilon_k}W(u_{\varepsilon_k}) \to 0 \quad \text{uniformly in this ball.}$$

Hence x_0 does not belong to Σ. This is a contradiction. $\square$

Lemma 10.6. *The blowing down limit* Σ *and* Ω_∞ *are unique.*

Proof. Assume for a sequence $\varepsilon_k \to 0$, the limit varifold V of u_{ε_k} has the form as in Proposition 10.3. We also assume that u_{ε_k} converges to χ_{Ω_∞}.

Let D_α, $1 \leq \alpha \leq K$, be the unbounded connected components of Ω^c. Since they are open convex sets, the blowing down limit of D_α (in the Hausdorff distance),

$$\lim_{\varepsilon \to 0} \varepsilon D_\alpha$$

exists, which we denote by D_α^∞. D_α^∞ is a convex cone. (It may have no interior points, depending on whether the opening angle of D_α is positive or zero.) Note that this limit is independent of $\varepsilon \to 0$.

Claim 1. $\Psi_\infty = 0$ on $\cup_\alpha \overline{D_\alpha^\infty}$.

This is because, for any compact set $K \subset \overline{D_\alpha^\infty}$ which is disjoint from the origin, it belongs to εD_α for all ε small.

This also implies $u_\infty = 0$ a.e. in $\cup_\alpha D_\alpha^\infty$.

Claim 2. $\partial D_\alpha^\infty \subset \Sigma$.

Otherwise, because both ∂D_α^∞ and Σ are cones,

$$\delta := dist(\partial D_\alpha^\infty \setminus B_{1/2}, \Sigma \setminus B_{1/2}) > 0.$$

For all ε_k small,

$$dist_H \left(\varepsilon_k \partial D_\alpha \cap (B_2 \setminus B_{1/2}), \partial D_\alpha^\infty \cap (B_2 \setminus B_{1/2})\right) \leq \delta/16. \tag{10.11}$$

Hence

$$dist \left(\varepsilon_k \partial D_\alpha \cap (B_2 \setminus B_{1/2}), \Sigma \cap (B_2 \setminus B_{1/2})\right) \geq \delta/2.$$

Take an $x_0 \in \partial D_\alpha^\infty \cap \partial B_{3/2}$. By the definition of Σ,

$$\lim_{\varepsilon_k \to 0} \int_{B_{\delta/2}(x_0)} \frac{\varepsilon_k}{2} |\nabla u_{\varepsilon_k}|^2 + \frac{1}{\varepsilon_k} W(u_{\varepsilon_k}) \chi_{\{u_{\varepsilon_k} > 0\}} = 0.$$

Hence by the Clearing Out Lemma (see [47, Proposition 3.1]), either $u_{\varepsilon_k} \equiv 0$ or $u_{\varepsilon_k} \geq 1 - \gamma$ in $B_{\delta/4}(x_0)$. By (10.11), $\varepsilon_k D_\alpha$ intersects $B_{\delta/8}(x_0)$. Thus the latter case does not happen. However, (10.11) also implies that $\{u_{\varepsilon_k} > 0\}$ intersects $B_{\delta/8}(x_0)$. This is a contradiction and the claim is proven.

Claim 3. $u_\infty = 1$ a.e. in $\mathbb{R}^2 \setminus \cup_\alpha \overline{D_\alpha^\infty}$.

By Proposition 9.3, there exists an $R_5 > 0$ such that $u > 0$ in $\mathcal{P} := \mathbb{R}^2 \setminus (B_{R_5} \cup \cup_\alpha D_\alpha)$. Because $\partial \mathcal{P} \setminus B_{R_5}$ consists only of finitely many unbounded smooth convex curves, the blowing down limit $\varepsilon \mathcal{P}$ has a unique limit $\mathcal{P}_\infty$ as $\varepsilon \to 0$, which is exactly $\mathbb{R}^2 \setminus \cup_\alpha D_\alpha^\infty$.

By Proposition 7.4, for any $\delta > 0$, there exists an $R > R_5$ such that, $u > 1 - \delta$ in $\mathcal{P} \setminus (\partial \mathcal{P})_R$, where

$$(\partial \mathcal{P})_R := \{x \in \mathcal{P} : \operatorname{dist}(x, \partial \mathcal{P}) \leq R\}.$$

Hence for any $\varepsilon > 0$, $u_\varepsilon > 1/2$ in $\varepsilon[\mathcal{P} \setminus (\partial\mathcal{P})_R]$. After passing to the limit and by noting the $L^1_{loc}(\mathbb{R}^2)$ convergence of u_ε, we see $u_\infty = 1$ a.e. in $\mathcal{P}_\infty$.

Together with the Clearing Out Lemma (see [47, Proposition 3.1]), the above claims imply that the interior point of $\mathcal{P}_\infty$ does not belong to Σ, and the interior of $\cup_\alpha D^\infty_\alpha$ does not belong to Σ either.

Combining Claim 1-3 we see $\Omega_\infty = \mathbb{R}^2 \setminus \cup_\alpha D^\infty_\alpha$ and $\Sigma = \cup_\alpha \partial D^\infty_\alpha$. By the uniqueness of D^∞_α we finish the proof. □

Proposition 10.7. *The density $n_\alpha = 1$ or 2. Moreover, if the opening angle of D_α is positive, the density on ∂D^∞_α equals 1, and if the opening angle of D_α equals 0, the density on ∂D^∞_α is 2.*

Proof. Assume $V = \sigma_0 \sum_\alpha n_\alpha [L_\alpha]$ where $n_\alpha \geq 1$ and $L_\alpha = \{re^{**}_\alpha : r \geq 0\}$. By the previous lemma, $\cup_\alpha L_\alpha = \Sigma$ is unique, i.e. independent of $\varepsilon \to 0$. Moreover, each L_α belongs to ∂D^∞_β for some β.

We have proved that $\Omega^c_\infty = \cup_\alpha D^\infty_\alpha$. By Lemma 8.5, if $\alpha \neq \beta$, ∂D^∞_α and ∂D^∞_β are disjoint outside the origin.

Since e^{**}_α are all distinct, for each α there is an open neighborhood U_α of $L_\alpha \cap (B_2 \setminus B_{1/2})$ such that these open sets are disjoint. By Theorem 1.6 and the proof of [44, Theorem 5], for any $\varepsilon > 0$ small, $\partial\{u_\varepsilon > 0\} \cap (B_2 \setminus B_{1/2})$ consists of exactly n_α smooth components of $\partial\{u_\varepsilon > 0\}$ in U_α, which can be represented by the graph of convex or concave functions defined on L_α with small Lipschitz constant.

First assume D^∞_α to be open. Then there are two unit vectors $e_+ \neq e_-$ such that $\partial D^\infty_\alpha = \{re_\pm : r \geq 0\}$. By the proof of the previous proposition, there exists an $R > 0$ such that, there are exactly two connected components of $\partial\Omega \setminus B_R$ asymptotic to $\{re_\pm : r \geq 0\}$ respectively. Hence the density of the varifold V on $\{re_\pm : r \geq 0\}$ is 1.

If D^∞_α is not open, it is a ray in the form $\{re : r \geq 0\}$ for some unit vector e. There exists an $R > 0$ such that, there are exactly two connected components of $\partial\Omega \setminus B_R$ asymptotic to $\{re : r \geq 0\}$. Hence the density of the varifold V on $\{re_\pm : r \geq 0\}$ is 2. □

Using this result we can prove Corollary 1.5.

Proof of Corollary 1.5. By Remark 10.4 and the previous proposition, we can assume the blowing down limit Σ is the x_1 axis, with density 1 or 2 on it. Since we have assumed u has two ends and each unbounded connected components of Ω^c gives two ends, there is only one unbounded connected component of Ω^c. The blowing limit of its boundary is the x_1 axis. Because it is convex, it can only be the half space. Then by Lemma 6.4, u is one dimensional. □

11 Refined asymptotics at infinity

In this section, we prove that u is exponentially close to its ends (one dimensional solutions) at infinity.

Let $\mathcal{C} = \{(x_1, x_2) : |x_2| < \lambda x_1, x_1 > R\}$ for some $\lambda > 0$ and $R > 0$. Assume that $u = 0$ in $\mathcal{C} \cap \{x_2 > f_+(x_1)\}$ and $\mathcal{C} \cap \{x_2 < f_-(x_1)\}$, where $f_\pm$ are convex (concave, respectively) functions defined on $[R, +\infty)$, satisfying

$$-\lambda x_1 < f_-(x_1) < f_+(x_1) < \lambda x_1.$$

Recall that Lemma 8.5 says

$$-\lambda \le \lambda_- := \lim_{x_1 \to +\infty} f_-'(x_1) < \lambda_+ := \lim_{x_1 \to +\infty} f_+'(x_1) \le \lambda.$$

By Proposition 9.3, there exists an $R > 0$ such that

$$u > 0 \quad \text{in } \{(x_1, x_2) : f_-(x_1) < x_2 < f_+(x_1), x_1 > R\}.$$

Hence after taking another larger R, the ray

$$\{(x_1, x_2) : x_2 = \frac{\lambda_- + \lambda_+}{2} x_1 + \frac{f_-(R) + f_+(R)}{2}, x_1 > R\}$$

is contained in $\{u > 0\}$.

After a rotation, we are in the following situation:

(H1) There are two positive constants $R > 0$ large and $\lambda > 0$.

(H2) There is a positive concave function $x_2 = f(x_1)$ defined on $[R, +\infty)$ such that

$$f'(x_1) > 0, \qquad \lim_{x_1 \to +\infty} f'(x_1) = 0.$$

In particular, as $x_1 \to +\infty$, $f(x_1) = o(x_1)$.

(H3) The domain $\mathcal{C} := \{(x_1, x_2) : f(x_1) < x_2 < \lambda x_1, x_1 > R\}$.

(H4) $u \in C^2(\overline{\mathcal{C}})$ and $u > 0$ in $\mathcal{C}$. Moreover,

$$\begin{cases} \Delta u = W'(u), & \text{in } \mathcal{C}, \\ u = 0, & \text{on } \{(x_1, x_2) : x_2 = f(x_1)\}, \\ |\nabla u| = \sqrt{2W(0)}, & \text{on } \{(x_1, x_2) : x_2 = f(x_1)\}. \end{cases}$$

For simplicity, we will also assume $u = 0$ below $\{x_2 = f(x_1)\}$. Then by the regularity theory in [1] and [28], f is smooth.

As in Part I, we want to emphasize that no stability condition is needed here.

This section is devoted to prove the following theorem, which also finishes the proof of Theorem 1.4.

Theorem 11.1. *There exists a constant t such that*

$$|f(x_1) - t| \le Ce^{-\frac{x_1}{C}},$$

and

$$|u(x_1, x_2) - g(x_2 - t)| \le Ce^{-\frac{x_1}{C}}.$$

First we note that

Lemma 11.2. *In $\overline{\mathcal{C}}$,*

$$1 - u(x_1, x_2) \le Ce^{-\frac{x_2 - f(x_1)}{C}}.$$

Proof. Because $0 < f'(x_1) < 1/2$, in $\mathcal{C}$ the distance to the curve $\{x_2 = f(x_1)\}$ is comparable to $x_2 - f(x_1)$.

Next by our assumptions, the translation of u along $\{x_2 = f(x_1)\}$ converges to $g(x_2)$ uniformly on compact sets of $\mathbb{R}^2$. Hence we can assume that, for some $L > 0$, $u(x_1, x_2) \ge 1 - \gamma$ in $\{(x_1, x_2) : f(x_1) + L < x_2 < \lambda x_1\}$. By the equation for u,

$$\Delta(1 - u) \ge c(1 - u) \quad \text{in } \{(x_1, x_2) : f(x_1) + L < x_2 < \lambda x_1\}.$$

The conclusion then follows from some standard methods, e.g. comparison with a sup solution. □

A direct corollary is

$$1 - u(x_1, x_2) \sim O(e^{-cx_1}) \quad \text{on } \{x_2 = \lambda x_1\}. \tag{11.1}$$

By Proposition 7.4, the limit at infinity of translations of u along $(x_1, f(x_1))$ is $g(x_2)$. The previous lemma then implies that

$$\lim_{x_1 \to +\infty} \sup_{x_2 \in \mathbb{R}} |u(x_1, x_2) - g(y - f(x_1))| = 0. \tag{11.2}$$

Another consequence of this exponential decay is:

Corollary 11.3. *In $\mathcal{C}$,*

$$|u_{x_1}(x_1, x_2)| + |u_{x_1 x_1}(x_1, x_2)| \le Ce^{-c(x_2 - f(x_1))}.$$

This follows from standard interior gradient estimates and boundary gradient estimates. (Note that $\partial\mathcal{C}$ is smooth with uniform C^3 bound.)

For application below, we also note the following Hamiltonian identity.

Proposition 11.4. *Let*

$$\rho(x_1) := \int_{f(x_1)}^{\lambda x_1} \frac{u_{x_2}^2 - u_{x_1}^2}{2} + W(u)dx_2, \quad x_1 > R.$$

Then

$$\rho(x_1) = \sigma_0 + O(e^{-cx_1}).$$

Proof. Differentiating in x_1 and integrating by parts give

$$\begin{aligned}
\rho'(x_1) &= \int_{f(x_1)}^{\lambda x_1} \left(u_{x_2}u_{x_2x_1} - u_{x_1}u_{x_1x_1} + W'(u)u_{x_1}\right) dx_2 \\
&\quad + \left(\frac{u_{x_2}^2 - u_{x_1}^2}{2} + W(u)\right)(x_1, f(x_1))f'(x_1) + O(e^{-cx_1}) \\
&= \int_{f(x_1)}^{\lambda x_1} \left(u_{x_2}u_{x_2x_1} + u_{x_1}u_{x_2x_2}\right) dx_2 + u_{x_2}(x_1, f(x_1))^2 f'(x_1) + O(e^{-cx_1}) \\
&= -u_{x_2}(x_1, f(x_1))u_{x_1}(x_1, f(x_1)) + u_{x_2}(x_1, f(x_1))^2 f'(x_1) + O(e^{-cx_1}) \\
&= O(e^{-cx_1}).
\end{aligned}$$

By the convergence of translations of u along $(x_1, f(x_1))$, Proposition 11.2 and Corollary 11.3, we get

$$\lim_{x_1\to+\infty} \rho(x_1) = \sigma_0,$$

and the conclusion follows. □

Let

$$v(x_1, x_2) := u(x_1, x_2) - g(x_2 - f(x_1)).$$

By Lemma 11.2 and (11.2),

$$\lim_{x_1\to+\infty} \|v\|_{L^2(0,\lambda x_1)} = 0. \tag{11.3}$$

In the following we denote

$$g_* := g(x_2 - f(x_1)).$$

For applications below, we note the following fact.

Lemma 11.5. *In $\mathcal{C}$, $u \le g_*$.*

Proof. Recall that the distance type function Ψ satisfies

$$\Psi_{x_2} \le 1, \quad \text{in } \mathcal{C}.$$

Because $\Psi = 0$ on $\{x_2 = f(x_1)\}$,

$$\Psi(x_1, x_2) \le x_2 - f(x_1), \quad \text{in } \mathcal{C}.$$

Since g is non-decreasing, we then obtain

$$u = g(\Psi) \le g(x_2 - f(x_1)), \quad \text{in } \mathcal{C}.$$

□

Similar to the calculation in [20, page 927], we have

$$\begin{aligned}
& \int_{f(x_1)}^{\lambda x_1} \left(\frac{u_{x_2}^2 - |g_*'|^2}{2} + W(u) - W(g_*) - \frac{u_{x_1}^2}{2} \right) \\
= & \int_{f(x_1)}^{\lambda x_1} \left[W(u) - W(g_*) - \frac{W'(u) + W'(g_*)}{2} (u - g_*) \right] \\
& + \frac{1}{2} \int_{f(x_1)}^{\lambda x_1} \left[(u - g_*) u_{x_1 x_1} - u_{x_1}^2 \right] + O(e^{-c x_1}) \\
= & \frac{1}{2} \int_{f(x_1)}^{\lambda x_1} \left[(u - g^*) u_{x_1 x_1} - u_{x_1}^2 \right] + o(\|v\|^2) + O(e^{-c x_1}).
\end{aligned}$$

Hence

$$\int_{f(x_1)}^{\lambda x_1} \left[u_{x_1 x_1} (u - g_*) - u_{x_1}^2 \right] = O(e^{-c x_1}) + o(\|v\|^2). \tag{11.4}$$

The following result says the second eigenvalue if g is positive.

Proposition 11.6. *For any $L > 0$ and $\phi \in H_0^1(0, L)$, there exists a constant $\mu > 0$ (independent of L) such that*

$$\int_0^L \phi'(t)^2 + W''(g(t))\phi(t)^2 dt \ge \mu \int_0^L \phi(t)^2 dt. \tag{11.5}$$

This can be proved by a contradiction argument.

Because $v(x_1, f(x_1)) = 0$, (11.5) applies to v, which gives

$$\begin{aligned}
& \int_{f(x_1)}^{\lambda x_1} \left[- (u - g_*)_{x_2 x_2} + W''(g_*) (u - g_*) \right] (u - g_*) \\
= & \int_{f(x_1)}^{\lambda x_1} \left| (u - g_*)_{x_2} \right|^2 + W''(g_*) (u - g_*)^2 + O(e^{-c x_1})
\end{aligned} \tag{11.6}$$

$$\geq \quad \mu\|v\|^2 + O(e^{-cx_1}).$$

Differentiating $\|v\|^2$ twice in x_1 leads to

$$\frac{1}{2}\frac{d}{dx_1}\|v\|^2 = \int_{f(x_1)}^{\lambda x_1} (u - g_*)\left[u_{x_1} + g_*' f'(x_1)\right] + O(e^{-cx_1}).$$

$$\begin{aligned}
\frac{1}{2}\frac{d^2}{dx_1^2}\|v\|^2 &= \int_{f(x_1)}^{\lambda x_1} u_{x_1}^2 + 2u_{x_1} g_*' f'(x_1) + \left|g_*'\right|^2 f'(x_1)^2 + u_{x_1x_1}(u - g^*) \\
&\quad - f'(x_1)^2 \int_{f(x_1)}^{\lambda x_1} g_*''(u - g_*) + f''(x_1)\int_{f(x_1)}^{\lambda x_1} g_*'(u - g_*) + O(e^{-cx_1}) \\
&= \int_{f(x_1)}^{\lambda x_1} \frac{16}{9} u_{x_1}^2 + 2u_{x_1} g_*' f'(x_1) + \left|g_*'\right|^2 f'(x_1)^2 + \frac{2}{9} u_{x_1x_1}(u - g^*) \quad \text{(by (11.4))} \\
&\quad - f'(x_1)^2 \int_{f(x_1)}^{\lambda x_1} g_*''(u - g_*) + f''(x_1)\int_{f(x_1)}^{\lambda x_1} g_*'(u - g_*) \\
&\quad + o(\|v\|^2) + O(e^{-cx_1}) \\
&= \int_{f(x_1)}^{\lambda x_1} \left[\frac{4}{3} u_{x_1}^2 + \frac{3}{4} g_*' f'(x_1)\right]^2 + \frac{2}{9} u_{x_1x_1}(u - g^*) \\
&\quad + f'(x_1)^2 \int_{f(x_1)}^{\lambda x_1} \left[\frac{7}{16}\left|g_*'\right|^2 - g_*''(u - g_*)\right] + f''(x_1)\int_{f(x_1)}^{\lambda x_1} g_*'(u - g_*) \\
&\quad + o(\|v\|^2) + O(e^{-cx_1}).
\end{aligned}$$

Note that the last integral is non-negative because $f'' \leq 0$, $g_*' \geq 0$ and $u - g_* \leq 0$ (see Lemma 11.5). We also have

$$f'(x_1)^2 \int_{f(x_1)}^{\lambda x_1} \left[\frac{7}{16}\left|g_*'\right|^2 - g_*''(u - g_*)\right] \geq 0, \tag{11.7}$$

because

$$\lim_{x_1 \to +\infty} \int_{f(x_1)}^{\lambda x_1} \frac{7}{16}\left|g_*'\right|^2 = \frac{7}{16}\sigma_0,$$

while

$$\int_{f(x_1)}^{\lambda x_1} g_*''(u - g_*) \leq \left[\int_{f(x_1)}^{\lambda x_1} \left|g_*''\right|^2\right]^{\frac{1}{2}} \|v\|,$$

which converges to 0 as $x_1 \to +\infty$, thanks to Corollary 11.3 and (11.3).

Thus

$$\frac{1}{2}\frac{d^2}{dx_1^2}\|v\|^2 \geq \frac{2}{9}\int_{f(x_1)}^{\lambda x_1} u_{x_1x_1}\,(u-g^*) + o(\|v\|^2) + O(e^{-cx_1}).$$

Next, similar to [20] we also have

$$\begin{aligned}
&\int_{f(x_1)}^{\lambda x_1} u_{x_1x_1}\,(u-g^*) \\
= &\int_{f(x_1)}^{\lambda x_1} (W'(u)-u_{x_2x_2})\,(u-g^*) \\
= &\int_{f(x_1)}^{\lambda x_1} [W'(u)-W'(g^*)-W''(g^*)\,(u-g^*)]\,(u-g^*) \\
&+\int_{f(x_1)}^{\lambda x_1} (g^{*\prime\prime}-u_{x_2x_2})\,(u-g^*)+W''(g^*)\,(u-g^*)^2 \\
\geq &\ (\mu+o(1))\,\|v\|^2 + O(e^{-cx_1}).
\end{aligned} \tag{11.8}$$

Here the last step is deduced from (11.6).

This then implies that

$$\frac{d^2}{dx_1^2}\|v\|^2 \geq c\|v\|^2 + O(e^{-cx_1}), \quad \text{for all } x_1 \text{ large.}$$

From this inequality and (11.3) we deduce that

$$\|v\|^2 \leq Ce^{-cx_1}, \quad \text{for all } x_1 \text{ large.}$$

By Corollary 11.3, for any $x_1 > R$,

$$\int_{f(x_1)}^{\lambda x_1} u_{x_1x_1}^2 \leq C.$$

Hence the Cauchy-Schwartz inequality gives

$$\int_{f(x_1)}^{\lambda x_1} u_{x_1x_1}\,(u-g_*) = O(e^{-cx_1}).$$

Then by (11.4),

$$\int_{f(x_1)}^{\lambda x_1} u_{x_1}^2 = O(e^{-cx_1}).$$

By the Cauchy-Schwartz inequality,

$$\int_{f(x_1)}^{\lambda x_1} \left|u_{x_1}\right| = O(e^{-cx_1}).$$

Integrating this in x_1 and noting that $u_{x_1} = 0$ below $\{x_2 = f(x_1)\}$, we get a function $u_\infty(x_2) = g(x_2 - t)$ for some constant t, such that

$$\int_{-\infty}^{+\infty} \left|u(x_1, x_2) - u_\infty(x_2)\right| dx_2 = O(e^{-cx_1}).$$

This can also be lifted to an estimate in $L^\infty(\mathbb{R})$ by the uniform Lipschitz bound on $u(x_1, x_2) - u_\infty(x_2)$.

For all x_1 large, in $[f(x_1), f(x_1) + 1]$,

$$u_{x_2} \geq \sqrt{2W(u)}/2,$$

which has a uniform positive lower bound. By this nondegeneracy property and a similar one for the one dimensional solution g, we deduce that

$$|f(x_1) - t| = O(e^{-cx_1}). \tag{11.9}$$

Another method to prove (11.9) is by noting (11.7), we in fact have

$$\frac{d^2}{dx_1^2}\|v\|^2 \geq cf'(x_1)^2.$$

Take a nonnegative function $\eta \in C_0^\infty((-2, 2))$ with $\eta \equiv 1$ in $(-1, 1)$. For any t large, testing the above inequality with $\eta(x_1 + t)$ and integrating by parts, we obtain

$$\int_{t-1}^{t+1} f'(x_1)^2 dx_1 = O(e^{-cx_1}).$$

Because f is concave and hence $f'(x_1)$ is non-increasing in x_1, this implies that

$$\left|f'(x_1)\right| = O(e^{-cx_1}),$$

and the exponential convergence of f follows.

Part III
Proof of Theorem 9.1

This part is devoted to a proof of Theorem 9.1. We first give some preliminary construction using the doubling lemma of Polácik-Quittner-Souplet [37] in Section 12. Then the proof is

divided into two cases. The first case is treated in Section 13 and the second one in Section 14-16. The second case is divided further into four subcases, the first two considered in Section 14, the third one in Section 15 and the last one in Section 16.

12 Reduction to a local estimate

Note that, by Lemma 7.4, H is bounded and

$$H(x) \to 0, \quad \text{as } x \to \infty. \tag{12.1}$$

With the extrinsic distance, $\partial\Omega$ is a complete metric space. Assume by the contrary, there exists $x_k \in \partial\Omega$ such that $H(x_k)|x_k| \geq 2k$. Because H is bounded on $\partial\Omega$, $|x_k| \to +\infty$. By the doubling lemma in [37], there exist $y_k \in \partial\Omega$ satisfying

$$H(y_k) \geq H(x_k), \qquad H(y_k)|y_k| \geq 2k,$$

$$H(x) \leq 2H(y_k) \qquad \text{for } \ x \in \partial\Omega \cap B_{kH(y_k)^{-1}}(y_k).$$

Because H is strictly positive and bounded (see (12.1)) on $\partial\Omega$, we must have $|y_k| \to +\infty$ and hence by (12.1),

$$\varepsilon_k := H(y_k) \to 0.$$

Define

$$u_k(x) := u(y_k + \varepsilon_k^{-1} x).$$

It satisfies

$$\begin{cases} \Delta u_k = \dfrac{1}{\varepsilon_k^2} W'(u_k), & \text{in } \{u_k > 0\}, \\ |\nabla u_k| = \dfrac{1}{\varepsilon_k}\sqrt{2W(0)}, & \text{on } \partial\{u_k > 0\}. \end{cases} \tag{12.2}$$

Because

$$|y_k| \geq 2k\varepsilon_k^{-1},$$

u_k is stable in $B_k(0)$. Moreover, by denoting H_k the curvature of $\partial\{u_k > 0\}$ (note that it is still positive), a rescaling gives

$$H_k(0) = 1, \quad \text{and} \quad H_k \leq 2 \quad \text{on } \partial\{u_k > 0\} \cap B_k(0). \tag{12.3}$$

By the latter curvature bound, for any $x \in \partial\{u_k > 0\} \cap B_{k-1}(0)$, the connected component of $\partial\{u_k > 0\} \cap B_{1/8}(x)$ containing x can be represented by the graph of a convex function with its $C^{1,1}$ norm bounded by 4.

Denote the connected component of $\partial\{u_k > 0\} \cap B_{1/8}(0)$ containing 0 by $\Gamma_{k,0}$, which we assume to be

$$\{(x_1, x_2) : x_2 = f_k(x_1)\}, \quad \text{for } x_1 \in (-1/8, 1/8),$$

where $f_k(0) = f_k'(0) = 0$. By (12.3), $-4 \le f_k'' \le 0$ and

$$f_k''(0) = -1. \tag{12.4}$$

Assume $\Gamma_{k,0}$ to be the boundary of a connected component of $\{u_k > 0\} \cap B_{1/8}(0)$, Ω_k. Without loss of generality, assume $u_k = 0$ in $B_{1/8}(0) \setminus \Omega_k$, that is, we ignore other connected components of $\{u_k > 0\} \cap B_{1/8}(0)$ except Ω_k.

We divide the proof into two cases.

Case 1. $\lim_{k\to+\infty} \operatorname{dist}(0, \partial\Omega_k \setminus \Gamma_{k,0}) > 0$.

By this assumption, there exists a constant $r > 0$ such that, in $B_r(0)$, $u_k = 0$ below $\{x_2 = f_k(x_1)\}$ and $u_k > 0$ above this curve.

We claim that there exists a constant $C(r)$ independent of k, such that

$$\int_{B_{r/2}(0)} \frac{\varepsilon_k}{2}|\nabla u_k|^2 + \frac{1}{\varepsilon_k} W(u_k)\chi_{\{u_k>0\}} \le C(r). \tag{12.5}$$

Indeed, by denoting $d(x)$ the distance to $\{u_k = 0\}$, using the convexity of $\{u_k = 0\}$ we get

$$\text{Length}(\{d(x) = t\} \cap B_{r/2}(0)) \le C(r), \quad \forall t \in (0, r/2). \tag{12.6}$$

Fix a large constant M. In $\{0 < d(x) < 2M\varepsilon_k\}$, because

$$\frac{\varepsilon_k}{2}|\nabla u_k(x)|^2 + \frac{1}{\varepsilon_k} W(u_k(x)) \le \frac{C}{\varepsilon_k},$$

and the area of $\{0 < d(x) < 2M\varepsilon_k\}$ is bounded by $O(\varepsilon_k)$ (by the coarea formula and (12.6)), we get

$$\int_{\{0<d(x)<2M\varepsilon_k\}\cap B_{r/2}(0)} \frac{\varepsilon_k}{2}|\nabla u_k(x)|^2 + \frac{1}{\varepsilon_k} W(u_k(x)) \le C(r). \tag{12.7}$$

On the other hand, by the stability of u_k and Theorem 1.6 (or by the uniform $C^{1,1}$ regularity of $\partial\{u_k > 0\}$), if we have chosen M large enough (it is independent of ε_k), we have

$$u_k(x) > \gamma, \quad \text{in } \{d(x) > M\varepsilon_k\}.$$

Hence by the differential inequality

$$\Delta\left[\frac{\varepsilon_k}{2}|\nabla u_k(x)|^2 + \frac{1}{\varepsilon_k} W(u_k(x))\right] \ge \frac{\kappa}{\varepsilon_k^2}\left[\frac{\varepsilon_k}{2}|\nabla u_k(x)|^2 + \frac{1}{\varepsilon_k} W(u_k(x))\right], \quad \text{in } \{u_k > \gamma\},$$

we get

$$\frac{\varepsilon_k}{2}|\nabla u_k(x)|^2+\frac{1}{\varepsilon_k}W(u_k(x))\leq\frac{C}{\varepsilon_k}e^{-\frac{d(x)}{C\varepsilon_k}},\quad\text{in }\{d(x)>2M\varepsilon_k\}\cap B_{r/2}(0).\tag{12.8}$$

By the co-area formula,

$$\begin{aligned}&\int_{\{d(x)>2M\varepsilon_k\}\cap B_{r/2}(0)}\frac{\varepsilon_k}{2}|\nabla u_k(x)|^2+\frac{1}{\varepsilon_k}W(u_k(x))\\=&\int_{2M\varepsilon_k}^{r/2}\left[\int_{\{d=t\}}\frac{\varepsilon_k}{2}|\nabla u_k(x)|^2+\frac{1}{\varepsilon_k}W(u_k(x))\right]dt\\\leq&\int_{2M\varepsilon_k}^{r/2}\left[\int_{\{d=t\}}\frac{C}{\varepsilon_k}e^{-\frac{d(x)}{C\varepsilon_k}}\right]dt\\\leq&\ C(r).\end{aligned}\tag{12.9}$$

Combining (12.7) with (12.9) gives (12.5).

By the $C^{1,1}$ bound on f_k, we can assume it converges to f_∞ in C^1. f_∞ is still a concave function satisfying $f_\infty(0)=f_\infty'(0)=0$. Using (12.5), we can apply the Hutchinson-Tonegawa theory (see [47]), which implies that $f_\infty\equiv 0$.

In Section 13 we will show that $f_k''(0)\to 0$. This is a contradiction with (12.4). Hence this case is impossible.

Case 2. $\lim_{k\to+\infty}\mathrm{dist}(0,\partial\Omega_k\setminus\Gamma_{k,0})=0$.

By this assumption, there exists $z_k\in\partial\Omega_k\setminus\Gamma_{k,0}$ such that $|z_k|\to 0$. Assume z_k attains $\mathrm{dist}(0,\partial\Omega_k\setminus\Gamma_{k,0})$. Recall that in $B_{1/8}(z_k)$, the connected component of $\partial\Omega_k$ passing through z_k is a graph and it is disjoint from $\Gamma_{k,0}$. Thus for all k large, in $B_{1/16}(0)$ this component has the form

$$\{x_2=\tilde{f}_k(x_1)\}:=\Gamma_{k,1},$$

where $\tilde{f}_k$ is a convex function with its $C^{1,1}$ norm bounded by 8.

Assume f_k and $\tilde{f}_k$ converge to f_∞ and $\tilde{f}_\infty$ in $C^1(-1/16,1/16)$. Then $f_\infty\leq\tilde{f}_\infty$, $f_\infty(0)=\tilde{f}_\infty(0)=0$.

We claim that there exists a constant $r>0$ independent of k such that $\{(x_1,x_2):f_k(x_1)<x_2<\tilde{f}_k(x_1),|x_1|<r\}$ is a connected component of $B_r(0)\cap\Omega_k$.

Assume by the contrary, there exists a third connected component of $B_r(0)\cap\partial\Omega_k$ lying between $\{x_2=f_k(x_1)\}$ and $\{x_2=\tilde{f}_k(x_1)$. Take an arbitrary point z on it. Let T be the tangent line of this component at z. Then this component can be represented by the graph of a convex function defined on an interval of this line, which contains z and has length at least 1/8. Because different components of $\partial\Omega_k\cap B_{1/2}(0)$ are disjoint, the tangent line at z must be almost parallel to the x_1-axis. Since z is arbitrary, this implies that this third component is also a graph on the x_1-axis, which is defined on $(-r,r)$. Since this curve lies

between $\{x_2 = f_k(x_1)\}$ and $\{x_2 = \tilde{f}_k(x_1)$, and $0 \in \{x_2 = f_k(x_1)\}$ and $z_k \in \{x_2 = \tilde{f}_k(x_1)$, there exists a point on this third component, which is closer to 0 than z_k. This is a contradiction with the choice of z_k and finishes the proof of this claim.

As in Case 1, by viewing $u_k = 0$ in $B_r(0) \setminus \{(x_1, x_2) : f_k(x_1) < x_2 < \tilde{f}_k(x_1), |x_1| < r\}$, u_k has uniformly bounded energy in $B_{r/2}(0)$. (This can also be proved by using the Hamiltonian identity, see the discussion of Subcase 2.1 in the proof of Theorem 10.1.) Then by Hutchinson-Tonegawa theory, $f_\infty = \tilde{f}_\infty = 0$.

In the following Theorem 14.1 we will show that $f_k''(0) \to 0$. This is a contradiction with (12.4). Hence this case is also impossible and we finish the proof of Theorem 9.1.

13 The first case

In this section we prove Case 1 in Section 12. We can consider the following setting. Let u_ε, $\varepsilon \to 0$, be a sequence of functions satisfying the following conditions.

A1) u_ε is a solution of

$$\begin{cases} \varepsilon \Delta u_\varepsilon = \dfrac{1}{\varepsilon} W'(u_\varepsilon), & \text{in } \{u_\varepsilon > 0\} \cap Q_1(0), \\ |\nabla u_\varepsilon| = \dfrac{1}{\varepsilon}\sqrt{2W(0)}, & \text{on } \partial\{u_\varepsilon > 0\} \cap Q_1(0). \end{cases}$$

Here $Q_1(0) = \{|x_1| < 1, |x_2| < 1\}$.

A2) There exists a constant $C > 0$ such that

$$\int_{Q_1(0)} \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon)\chi_{\{u_\varepsilon > 0\}} \le C.$$

A3) u_ε satisfies the Modica inequality. Hence $\partial\{u_\varepsilon > 0\}$ is convex.

A4) There exists a concave function f_ε with $f_\varepsilon(0) = f_\varepsilon'(0) = 0$, such that

$$\{u_\varepsilon > 0\} \cap Q_1(0) = \{(x_1, x_2) : x_2 > f_\varepsilon(x_1)\}.$$

A5) f_ε are uniformly bounded in $C^{1,1}(-1,1)$ and $f_\varepsilon \to 0$ uniformly on $(-1,1)$.

Note that u_ε has only one connected component of free boundaries. Hence by the main result in [47] (see also [46]), under the above hypothesis, f_ε is automatically uniformly bounded in $C^{1,\alpha}_{loc}((-1,1))$ for some $\alpha \in (0,1)$. (If u_ε is stable, a $C^{1,1/2}$ regularity can also be proved directly, see [44]. But note that in the above **A1-5)** no stability condition is assumed.) Combining this with Hutchinson-Tonegawa theory, we get $f_\varepsilon \to 0$ in $C^1_{loc}(-1,1)$. As a consequence, the uniform $C^{1,1}$ bound on f_ε in **A5)** is not necessary.

The main result in this section is

Theorem 13.1. *Under the above hypothesis* **A1)-A5)**,

$$\lim_{\varepsilon\to 0} f_\varepsilon''(0) = 0.$$

This theorem implies the first case introduced in Section 12 is impossible. The remaining part of this section will be devoted to the proof of this theorem.

The main tool to prove Theorem 13.1 is the following decay estimate, which says f_ε decays faster than quadratically at 0.

Lemma 13.2. *For any $r \in (0, 1/2)$, either*

$$\max\{\left|f_\varepsilon(r)\right|, \left|f_\varepsilon(-r)\right|\} = o(\varepsilon),$$

or

$$\max\left\{\left|f_\varepsilon\left(\frac{r}{2}\right)\right|, \left|f_\varepsilon\left(-\frac{r}{2}\right)\right|\right\} \le \frac{1}{8}\max\left\{\left|f_\varepsilon(r)\right|, \left|f_\varepsilon(-r)\right|\right\}.$$

Remark 13.3. *There is a similar statement for $\int \left|f_\varepsilon''\right|$. It says either*

$$\int_{-r/2}^{r/2} \left|f_\varepsilon''(x_1)\right| dx_1 = o\left(\frac{\varepsilon}{r}\right),$$

or

$$\int_{-r/2}^{r/2} \left|f_\varepsilon''(x_1)\right| dx_1 \le \frac{1}{8}\int_{-r}^{r} \left|f_\varepsilon''(x_1)\right| dx_1.$$

The proof is similar.

Before going into the proof, let us first present several technical lemmas.

Lemma 13.4. *In $\{u_\varepsilon > 0\}$,*

$$1 - u_\varepsilon(x_1, x_2) \le C e^{-\frac{x_2 - f_\varepsilon(x_1)}{C\varepsilon}}.$$

This is the scaled version of Lemma 11.2. We also have similar exponential decay on the gradients of u_ε.

Let Ψ_ε be the distance type function defined by

$$u_\varepsilon = g\left(\frac{\Psi_\varepsilon}{\varepsilon}\right).$$

Lemma 13.5. *For any $L > 0$, we have*

$$\left|\frac{\partial \Psi_\varepsilon}{\partial x_1}\right| + \left|\frac{\partial \Psi_\varepsilon}{\partial x_2} - 1\right| = o_\varepsilon(1), \quad \text{in } \{f_\varepsilon(x_1) < x_2 < f_\varepsilon(x_1) + L\varepsilon, |x_1| < 3/4\},$$

where $o_\varepsilon(1)$ converges to 0 as $\varepsilon \to 0$, uniformly in $\overline{\{f_\varepsilon(x_1) < x_2 < f_\varepsilon(x_1) + L\varepsilon\}}$.

Proof. For any $t_\varepsilon \in (-3/4, 3/4)$, let

$$\bar{u}^\varepsilon(x_1, x_2) := u_\varepsilon(t_\varepsilon + \varepsilon x_1, f_\varepsilon(t_\varepsilon) + \varepsilon x_2),$$

which is a solution of (1.6) in $B_{\varepsilon^{-1}/4}(0)$. $\partial\{\bar{u}^\varepsilon > 0\}$ is the curve

$$\left\{ x_2 = \frac{1}{\varepsilon}\left[f_\varepsilon(t_\varepsilon + \varepsilon x_1) - f_\varepsilon(t_\varepsilon)\right] \right\}.$$

Note that by **A5)**,

$$\frac{1}{\varepsilon}\left[f_\varepsilon(t_\varepsilon + \varepsilon x_1) - f_\varepsilon(t_\varepsilon)\right] \to 0 \quad \text{in } C^1_{loc}(\mathbb{R}). \tag{13.1}$$

Since $\bar{u}^\varepsilon$ satisfies the Modica inequality, it is globally Lipschitz. Assume it converges to $\bar{u}$ in $C_{loc}(\mathbb{R}^2)$. Then similar to Lemma 7.4, $\bar{u}$ is a solution of (1.6) in $\mathbb{R}^2$, with its free boundary consisting of a straight line. By Lemma 6.4 and (13.1), $\bar{u} = g(x_2)$.

Let $\bar{\Psi}^\varepsilon$ be defined through the relation

$$\bar{u}^\varepsilon = g(\bar{\Psi}^\varepsilon).$$

By the convergence of $\bar{u}^\varepsilon$, $\bar{\Psi}^\varepsilon$ converges to x_2 in C^1 manner.

By noting that

$$\bar{\Psi}^\varepsilon(x_1, x_2) = \frac{1}{\varepsilon}\Psi_\varepsilon(t_\varepsilon + \varepsilon x_1, f_\varepsilon(t_\varepsilon) + \varepsilon x_2),$$

the conclusion of this lemma follows. $\square$

Consider the kth order momentum

$$M^k_\varepsilon(x_1) := \int_{f_\varepsilon(x_1)}^1 x_2^k \left[\frac{1}{\varepsilon}W(u_\varepsilon) + \frac{\varepsilon}{2}\left(\left|\frac{\partial u_\varepsilon}{\partial x_2}\right|^2 - \left|\frac{\partial u_\varepsilon}{\partial x_1}\right|^2\right)\right] dx_2, \quad k = 0, 1, \cdots.$$

In the following, for simplicity we denote

$$\omega_\varepsilon := \frac{1}{\varepsilon}W(u_\varepsilon) + \frac{\varepsilon}{2}\left(\left|\frac{\partial u_\varepsilon}{\partial x_2}\right|^2 - \left|\frac{\partial u_\varepsilon}{\partial x_1}\right|^2\right).$$

Lemma 13.6. *In* $(-1, 1)$, *we have*

$$\frac{d}{dx_1}M^0_\varepsilon(x_1) = O(e^{-c\varepsilon^{-1}}), \tag{13.2}$$

$$\frac{d^2}{dx_1^2}M^1_\varepsilon(x_1) = O(e^{-c\varepsilon^{-1}}). \tag{13.3}$$

Proof. The proof is similar to the derivation of the Hamiltonian identity, by differentiating in x_1 and integrating by parts. By the free boundary condition, those terms on the free boundary appearing in this procedure cancel with each other. By Lemma 13.4, those terms on $\{x_2 = 1\}$ appearing in this procedure is of the order $O(e^{-c\varepsilon^{-1}})$. $\square$

Lemma 13.7. *There exists a constant E_ε satisfying*

$$\lim_{\varepsilon\to 0} E_\varepsilon = 0,$$

such that

$$M^0(x_1) \equiv \sigma_0 + E_\varepsilon + O(e^{-c\varepsilon^{-1}}) \quad \text{in } (-1,1). \tag{13.4}$$

Moreover, we have

$$M^1_\varepsilon(x_1) = (\sigma_0 + E_\varepsilon)\, f_\varepsilon(x_1) + O(\varepsilon) \quad \text{in } (-1,1). \tag{13.5}$$

Proof. (13.4) follows from the Hamiltonian identity (13.2) and Hutchinson-Tonegawa theory.

Decompose $M^1_\varepsilon(x_1)$ as follows:

$$M^1_\varepsilon(x_1) = \int_{f_\varepsilon(x_1)}^1 [x_2 - f_\varepsilon(x_1)]\, \omega_\varepsilon dx_2 + f_\varepsilon(x_1)\int_{f_\varepsilon(x_1)}^1 \omega_\varepsilon dx_2. \tag{13.6}$$

By Lemma 13.4, we have

$$\begin{aligned}
\int_{f_\varepsilon(x_1)}^1 [x_2 - f_\varepsilon(x_1)]\, \omega_\varepsilon dx_2 &\le C\int_{f_\varepsilon(x_1)}^1 [x_2 - f_\varepsilon(x_1)]\, e^{-c\frac{x_2 - f_\varepsilon(x_1)}{\varepsilon}} dx_2 \\
&\le C\varepsilon \int_0^{2\varepsilon^{-1}} x_2 e^{-cx_2} dx_2 \\
&\le C\varepsilon.
\end{aligned}$$

Substituting this and (13.4) into (13.6) we get (13.5). $\square$

Proof of Lemma 13.2. Assume by the contrary, there exists a constant $\lambda > 0$, a sequence of functions u_ε and f_ε satisfying the hypothesis **A1)-A5)**, and a sequence of $r_\varepsilon \in (0, 1/2)$ such that,

$$h_\varepsilon := \max\{\left|f_\varepsilon(r_\varepsilon)\right|, \left|f_\varepsilon(-r_\varepsilon)\right|\} \ge \lambda\varepsilon, \tag{13.7}$$

and

$$\max\left\{\left|f_\varepsilon\left(\frac{r_\varepsilon}{2}\right)\right|, \left|f_\varepsilon\left(-\frac{r_\varepsilon}{2}\right)\right|\right\} > \frac{1}{8}\max\left\{\left|f_\varepsilon(r_\varepsilon)\right|, \left|f_\varepsilon(-r_\varepsilon)\right|\right\}. \tag{13.8}$$

We will derive a contradiction from these two assumptions.

Let

$$v_\varepsilon(x_1, x_2) := u_\varepsilon(r_\varepsilon x_1, h_\varepsilon x_2), \quad \tilde{f}_\varepsilon(x_1) := \frac{1}{h_\varepsilon} f_\varepsilon(r_\varepsilon x_1).$$

We have

1. v_ε is a solution of

$$\begin{cases} r_\varepsilon^{-2}\dfrac{\partial^2 v_\varepsilon}{\partial x_1^2}+h_\varepsilon^{-2}\dfrac{\partial^2 v_\varepsilon}{\partial x_2^2}=\varepsilon^{-2}W'(v_\varepsilon), & \text{in } \{v_\varepsilon>0\}\cap(-r_\varepsilon^{-1},r_\varepsilon^{-1})\times(-h_\varepsilon^{-1},h_\varepsilon^{-1}),\\ \dfrac{\varepsilon h_\varepsilon}{2r_\varepsilon^2}\Big|\dfrac{\partial v_\varepsilon}{\partial x_1}\Big|^2+\dfrac{\varepsilon}{2h_\varepsilon}\Big|\dfrac{\partial v_\varepsilon}{\partial x_2}\Big|^2=\dfrac{h_\varepsilon}{\varepsilon}W(0), & \text{on } \partial\{v_\varepsilon>0\}.\end{cases}$$

2. v_ε satisfies the Modica inequality

$$\frac{\varepsilon h_\varepsilon}{2r_\varepsilon^2}\Big|\frac{\partial v_\varepsilon}{\partial x_1}\Big|^2+\frac{\varepsilon}{2h_\varepsilon}\Big|\frac{\partial v_\varepsilon}{\partial x_2}\Big|^2\le\frac{h_\varepsilon}{\varepsilon}W(v_\varepsilon),\quad \text{in } \{v_\varepsilon>0\}. \tag{13.9}$$

3. $\tilde{f}_\varepsilon$ is a concave function satisfying $\tilde{f}_\varepsilon(0)=\tilde{f}_\varepsilon'(0)=0$. Moreover,

$$\{v_\varepsilon>0\}=\{(x_1,x_2):x_2>\tilde{f}_\varepsilon(x_1)\}.$$

4. We have

$$\max\Big\{\big|\tilde{f}_\varepsilon(1)\big|,\big|\tilde{f}_\varepsilon(-1)\big|\Big\}=1, \tag{13.10}$$

and

$$\max\left\{\Big|\tilde{f}_\varepsilon\left(\frac{1}{2}\right)\Big|,\Big|\tilde{f}_\varepsilon\left(-\frac{1}{2}\right)\Big|\right\}\ge\frac{1}{8}. \tag{13.11}$$

By the last two conditions, we can assume $\tilde{f}_\varepsilon$ converges to a limit $\tilde{f}$ locally uniformly in $(-1,1)$, which is a concave function satisfying $\tilde{f}(0)=0$, $\tilde{f}\le 0$,

$$\sup_{x_1\in(-1,1)}|\tilde{f}(x_1)|\le 1,$$

and

$$\max\{\Big|\tilde{f}\left(\frac{1}{2}\right)\Big|,\Big|\tilde{f}\left(-\frac{1}{2}\right)\Big|\}\ge\frac{1}{8}. \tag{13.12}$$

The last one implies that $\tilde{f}$ is nonzero in $(-3/4,3/4)$.

There are two cases depending on whether $h_\varepsilon\gg\varepsilon$ or $h_\varepsilon\sim\varepsilon$.

Case 1. $\lim_{\varepsilon\to0}\frac{h_\varepsilon}{\varepsilon}=+\infty$.

By (13.5), this implies that

$$\tilde{f}_\varepsilon(x_1)=\frac{1}{(\sigma_0+o(1))h_\varepsilon}M_\varepsilon^1(r_\varepsilon x_1)+o(1).$$

Because $M_\varepsilon^1(r_\varepsilon x_1)$ is a linear function with an $O(e^{-c\varepsilon^{-1}})$ error, we see $\tilde{f}_\varepsilon(x_1)$ is a linear function with an $O(h_\varepsilon^{-1}e^{-c\varepsilon^{-1}})=o(1)$ error. After passing to the limit, we get that $\tilde{f}$ is a

linear function. Because $\tilde{f}(0)=0$ and $\tilde{f}\leq 0$, $\tilde{f}\equiv 0$ in $(-1,1)$. This contradicts (13.12). Hence this case is impossible.

Case 2. There exists a constant $\tau\in(0,+\infty)$ such that $\lim_{\varepsilon\to 0}\frac{h_\varepsilon}{\varepsilon}=\tau$.

Let

$$\tilde{\Psi}_\varepsilon(x_1,x_2):=\frac{1}{h_\varepsilon}\Psi_\varepsilon(r_\varepsilon x_1,h_\varepsilon x_2),$$

which satisfies

$$v_\varepsilon(x_1,x_2)=g\left(\frac{h_\varepsilon\tilde{\Psi}_\varepsilon}{\varepsilon}\right). \tag{13.13}$$

By Lemma 13.5, for any $L>0$, in $\{(x_1,x_2):\tilde{f}_\varepsilon(x_1)<x_2<\tilde{f}_\varepsilon(x_1)+L,-1<x_1<1\}$ (after rescaling back to u_ε, it belongs to $\{f_\varepsilon(x_1)<x_2<f_\varepsilon(x_1)+Lh_\varepsilon\}$),

$$\frac{\partial\tilde{\Psi}_\varepsilon}{\partial x_2}\to 1,\qquad \text{uniformly.} \tag{13.14}$$

Because $\tilde{\Psi}_\varepsilon=0$ on $\{x_2=\tilde{f}_\varepsilon(x_1)\}$ and $\tilde{f}_\varepsilon$ converges to $\tilde{f}$ uniformly, for any $L>0$,

$$\tilde{\Psi}_\varepsilon\to\left(x_2-\tilde{f}(x_1)\right)_+,\qquad \text{uniformly in } (-3/4,3/4)\times(-L,L). \tag{13.15}$$

By (13.13), we also have the uniform convergence

$$v_\varepsilon\to g\left(\tau(x_2-\tilde{f}(x_1))\right),\qquad \text{in } (-3/4,3/4)\times(-L,L). \tag{13.16}$$

For any $\delta>0$, by (13.4), there exists an $L>0$ such that

$$\int_{\tilde{f}_\varepsilon(x_1)+L}^{h_\varepsilon^{-1}}x_2\tilde{\omega}_\varepsilon(x_1,x_2)\leq\delta,$$

where

$$\tilde{\omega}_\varepsilon:=\frac{h_\varepsilon}{\varepsilon}W(v_\varepsilon)+\frac{\varepsilon}{2h_\varepsilon}\left|\frac{\partial v_\varepsilon}{\partial x_2}\right|^2-\frac{\varepsilon h_\varepsilon}{2r_\varepsilon^2}\left|\frac{\partial v_\varepsilon}{\partial x_1}\right|^2.$$

Therefore

$$\begin{aligned}\widetilde{M}_\varepsilon^1(x_1) &:= \int_{\tilde{f}_\varepsilon(x_1)}^{h_\varepsilon^{-1}}x_2\tilde{\omega}_\varepsilon(x_1,x_2)dx_2 \\ &= \int_{\tilde{f}_\varepsilon(x_1)}^{\tilde{f}_\varepsilon(x_1)+L}x_2\tilde{\omega}_\varepsilon(x_1,x_2)dx_2+O(\delta).\end{aligned} \tag{13.17}$$

Next, we have

$$\begin{aligned}
&\int_{\tilde{f}_\varepsilon(x_1)}^{\tilde{f}_\varepsilon(x_1)+L} x_2\tilde{\omega}_\varepsilon(x_1,x_2)dx_2 \\
= &\int_{\tilde{f}_\varepsilon(x_1)}^{\tilde{f}_\varepsilon(x_1)+L} x_2\frac{h_\varepsilon}{\varepsilon}W(v_\varepsilon)\left[1+\Big|\frac{\partial\tilde{\Psi}_\varepsilon}{\partial x_2}\Big|^2-\frac{h_\varepsilon^2}{r_\varepsilon^2}\Big|\frac{\partial\tilde{\Psi}_\varepsilon}{\partial x_1}\Big|^2\right]dx_2,
\end{aligned}$$

where by Lemma 13.5, we have

$$\lim_{\varepsilon\to 0}\int_{\tilde{f}_\varepsilon(x_1)}^{\tilde{f}_\varepsilon(x_1)+L}\frac{h_\varepsilon}{\varepsilon}W(v_\varepsilon)\frac{h_\varepsilon^2}{r_\varepsilon^2}\Big|\frac{\partial\tilde{\Psi}_\varepsilon}{\partial x_1}\Big|^2dx_2=0,\quad \forall x_1\in(-3/4,3/4).$$

Then by (13.14), we get

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\tilde{f}_\varepsilon(x_1)}^{\tilde{f}_\varepsilon(x_1)+L} x_2\tilde{\omega}_\varepsilon(x_1,x_2)dx_2 \\
= &\lim_{\varepsilon\to 0}\int_{\tilde{f}_\varepsilon(x_1)}^{\tilde{f}_\varepsilon(x_1)+L} x_2\frac{2h_\varepsilon}{\varepsilon}W(v_\varepsilon) \\
= &\,2\tau\int_{\tilde{f}(x_1)}^{\tilde{f}(x_1)+L} x_2W\left(g\left(\tau(x_2-\tilde{f}(x_1))\right)\right).
\end{aligned}$$

Substituting this into (13.17) we see

$$\begin{aligned}
\lim_{\varepsilon\to 0}\widetilde{M}_\varepsilon^1(x_1) &= 2\tau\int_{\tilde{f}(x_1)}^{\tilde{f}(x_1)+L} x_2W\left(g\left(\tau(x_2-\tilde{f}(x_1))\right)\right)+O(\delta) \\
&= 2\tau\tilde{f}(x_1)\int_0^L W\left(g\left(\tau x_2\right)\right)dx_2+2\tau\int_0^L x_2W\left(g\left(\tau x_2\right)\right)dx_2+O(\delta) \\
&= 2\tau\tilde{f}(x_1)\int_0^{+\infty} W\left(g\left(\tau x_2\right)\right)dx_2+2\tau\int_0^{+\infty} x_2W\left(g\left(\tau x_2\right)\right)dx_2+O(\delta),
\end{aligned}$$

where in the last step we have used the exponential convergence of g at infinity. Since δ can be arbitrarily small, this implies that

$$\lim_{\varepsilon\to 0}\widetilde{M}_\varepsilon^1(x_1)=2\tau\tilde{f}(x_1)\int_0^{+\infty}W(g\left(\tau x_2\right))dx_2+2\tau\int_0^{+\infty}x_2W(g\left(\tau x_2\right))dx_2.$$

Because

$$\widetilde{M}_\varepsilon^1(x_1)=\frac{1}{h_\varepsilon}M_\varepsilon^1(r_\varepsilon x_1)$$

is a linear function with an $O(\varepsilon^{-1}e^{-c\varepsilon^{-1}})=o(1)$ error, the left hand side is linear in x_1. Thus $\tilde{f}$ is also linear. Then arguing as in Case 1 we get a contradiction. $\square$

With this decay estimate at hand we can prove Theorem 13.1.

Proof of Theorem 13.1. Assume by the contrary, there exists a constant $\lambda > 0$ such that,

$$\liminf_{\varepsilon\to 0} \left|f_\varepsilon''(0)\right| \geq \lambda. \tag{13.18}$$

We have

Claim. For any $r > \varepsilon$, the following holds

$$\max\left\{\left|f_\varepsilon(r)\right|, \left|f_\varepsilon(-r)\right|\right\} \geq \frac{\lambda}{16C_H}\varepsilon, \tag{13.19}$$

where C_H is the constant in Proposition 7.5. (This proposition holds because we have the $C^{1,1}$ regularity of the free boundary, and hence higher regularity by the main result in [28].)

Indeed, if (13.19) does not hold, then by the concavity of f_ε and the fact that $f_\varepsilon(0) = f_\varepsilon'(0) = 0$,

$$\max_{|x_1|<r/2} \left|f_\varepsilon'(x_1)\right| \leq \frac{\lambda}{4C_H}\frac{\varepsilon}{r}.$$

Let

$$f_{\varepsilon,r}(x_1) := \frac{r}{\varepsilon} f_\varepsilon\left(\frac{\varepsilon}{r}x_1\right).$$

By the concavity and the above gradient bound,

$$\begin{aligned}\int_{-1}^{1} \left|f_{\varepsilon,r}''(x_1)\right| dx_1 &= f_{\varepsilon,r}'(-1) - f_{\varepsilon,r}'(1)\\ &\leq \frac{\lambda}{2C_H}\frac{\varepsilon}{r}.\end{aligned}$$

Then by Proposition 7.5,

$$\left|f_{\varepsilon,r}''(0)\right| \leq \frac{\lambda}{2}\frac{\varepsilon}{r}.$$

Rescaling back this says

$$\left|f_\varepsilon''(0)\right| \leq \frac{\lambda}{2}.$$

This is a contradiction with (13.18) and the claim follows.

By this claim, applying Lemma 13.2 to $r_j = 2^{-j}$ gives, for any $1 \leq j \leq \frac{|\log\varepsilon|}{\log 2}$,

$$\max\left\{\left|f_\varepsilon(2^{-j-1})\right|, \left|f_\varepsilon(-2^{-j-1})\right|\right\} \leq \frac{1}{8}\max\left\{\left|f_\varepsilon(2^{-j})\right|, \left|f_\varepsilon(-2^{-j})\right|\right\}.$$

Iterating this we get

$$\max\left\{\left|f_\varepsilon(2^{-j})\right|, \left|f_\varepsilon(-2^{-j})\right|\right\} \leq C8^{-j}, \quad \forall 1 \leq j \leq \frac{|\log\varepsilon|}{\log 2}.$$

By the concavity of f_ε and the fact that $f_\varepsilon(0) = f'_\varepsilon(0) = 0$, this implies

$$\max\left\{\left|f_\varepsilon(\varepsilon)\right|, \left|f_\varepsilon(-\varepsilon)\right|\right\} \le C\varepsilon^3.$$

Then following the proof of (13.19), we get

$$\left|f''_\varepsilon(0)\right| \le C\varepsilon,$$

which is a contradiction with (13.18) if ε is small enough. This completes the proof of this theorem. □

14 The second case

In this section we consider Case 2 introduced in Section 12. Thus let u_ε, $\varepsilon \to 0$, be a sequence of functions, satisfying the following conditions.

B1) u_ε is a solution of

$$\begin{cases} \varepsilon\Delta u_\varepsilon = \dfrac{1}{\varepsilon}W'(u_\varepsilon), & \text{in } \{u_\varepsilon > 0\} \cap \mathcal{C}_1(0), \\ |\nabla u_\varepsilon| = \dfrac{1}{\varepsilon}\sqrt{2W(0)}, & \text{on } \partial\{u_\varepsilon > 0\} \cap \mathcal{C}_1(0). \end{cases}$$

Here $\mathcal{C}_1(0) = \{|x_1| < 1, -\infty < x_2 < +\infty\}$.

B2) There exists a constant $C > 0$ such that

$$\int_{\mathcal{C}_1(0)} \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\chi_{\{u_\varepsilon>0\}} < C.$$

B3) u_ε satisfies the Modica inequality. Hence $\partial\{u_\varepsilon > 0\}$ is convex.

B4) There exists a concave function f^-_ε and a convex function f^+_ε with $f^-_\varepsilon(0) = f^{-\prime}_\varepsilon(0) = 0$ and $f^+_\varepsilon > f^-_\varepsilon$, such that

$$\{u_\varepsilon > 0\} \cap \mathcal{C}_1(0) = \{(x_1, x_2) : f^-_\varepsilon(x_1) < x_2 < f^+_\varepsilon(x_1)\}.$$

B5) $f^\pm_\varepsilon$ are uniformly bounded in $C^{1,1}(-1, 1)$.

B6) $f^\pm_\varepsilon \to 0$ in $C^1(-1, 1)$.

The main result in this section is similar to Theorem 13.1.

Theorem 14.1. *Under the above hypothesis* **B1)-B6)***,*

$$\lim_{\varepsilon\to 0} f''_\varepsilon(0) = 0.$$

This theorem implies the second case introduced in Section 12 is also impossible and the proof of Theorem 9.1 is thus complete.

14.1 Several technical results

First let us present some technical results.

Lemma 14.2. *For any $\delta > 0$, as $\varepsilon \to 0$,*

$$\inf_{-1+\delta \le x_1 \le 1-\delta} \frac{f_\varepsilon^+(x_1) - f_\varepsilon^-(x_1)}{\varepsilon} \to +\infty.$$

In the remaining part of this paper, various estimates will only hold (uniformly) in the interior, say $\mathcal{C}_{1-\delta}$, with the constant in the estimate depending on δ. Unless otherwise stated, this will not be stated explicitly.

Proof. For any $t_\varepsilon \in [1-\delta, 1-\delta]$, let

$$\bar{u}^\varepsilon(x_1, x_2) := u_\varepsilon(t_\varepsilon + \varepsilon x_1, f_\varepsilon^-(t_\varepsilon) + \varepsilon x_2),$$

which is a solution of (1.6) in $\mathcal{C}_{\delta\varepsilon^{-1}}(0)$. $\partial\{\bar{u}^\varepsilon > 0\}$ consists of two curves

$$\left\{ x_2 = \frac{1}{\varepsilon} f_\varepsilon^\pm(\varepsilon x_1) \right\},$$

which converge to two lines in C^1 manner (by **B6)**). Note that

$$\frac{1}{\varepsilon} f_\varepsilon^-(\varepsilon x_1) \to 0 \quad \text{in } C^1_{loc}(\mathbb{R}). \tag{14.1}$$

Since $\bar{u}^\varepsilon$ satisfies the Modica inequality, it is globally Lipschitz. Assume it converges to $\bar{u}$ in $C_{loc}(\mathbb{R}^2)$. Then similar to Lemma 7.4, $\bar{u}$ is a solution of (1.6) in $\mathbb{R}^2$, with its free boundary consisting of straight lines. By Proposition 6.4 and (14.1), $\bar{u} = g(x_2)$. In particular, for any $L > 0$, $\bar{u}^\varepsilon > 0$ in

$$\left\{ \frac{1}{\varepsilon} f_\varepsilon^-(\varepsilon x_1) < x_2 < \frac{1}{\varepsilon} f_\varepsilon^-(\varepsilon x_1) + L \right\}.$$

This clearly implies the conclusion of this lemma. $\square$

Let Ψ_ε be the distance type function defined by

$$u_\varepsilon = g\left(\frac{\Psi_\varepsilon}{\varepsilon}\right).$$

The Modica inequality implies that $|\nabla \Psi_\varepsilon| \le 1$, see [46, Lemma A.1].

Lemma 14.3. *For any $L > 0$, we have*

$$\left|\frac{\partial \Psi_\varepsilon}{\partial x_1}\right| + \left|\frac{\partial \Psi_\varepsilon}{\partial x_2} - 1\right| = o_\varepsilon(1), \quad \text{in } \{f_\varepsilon^-(x_1) < x_2 < f_\varepsilon^-(x_1) + L\varepsilon\},$$

and

$$\left|\frac{\partial \Psi_\varepsilon}{\partial x_1}\right| + \left|\frac{\partial \Psi_\varepsilon}{\partial x_2} + 1\right| = o_\varepsilon(1), \quad \text{in } \{f_\varepsilon^+(x_1) - L\varepsilon < x_2 < f_\varepsilon^+(x_1)\},$$

where $o_\varepsilon(1)$ converges to 0 as $\varepsilon \to 0$, uniformly in the above two domains.

Proof. Consider the function $\bar{u}^\varepsilon$ introduced in the proof of the previous lemma. Let $\bar{\Psi}^\varepsilon$ be defined through the relation

$$\bar{u}^\varepsilon = g(\bar{\Psi}^\varepsilon).$$

By the convergence of $\bar{u}^\varepsilon$ established in the proof of the previous lemma, $\bar{\Psi}^\varepsilon$ converges to x_2 in C^1 manner in $\{\varepsilon^{-1} f_\varepsilon^-(x_1) < x_2 < \varepsilon^{-1} f_\varepsilon^-(x_1) + L\}$.

By noting that in this domain

$$\bar{\Psi}^\varepsilon(x_1, x_2) = \frac{1}{\varepsilon}\Psi_\varepsilon(t_\varepsilon + \varepsilon x_1, f_\varepsilon^-(t_\varepsilon) + \varepsilon x_2),$$

the first conclusion of this lemma follows. The second one can be proved by the same method. $\square$

14.2 Momentum

Consider the kth order momentum

$$M_\varepsilon^k(x_1) := \int_{f_\varepsilon^-(x_1)}^{f_\varepsilon^+(x_1)} x_2^k \left[\frac{1}{\varepsilon}W(u_\varepsilon) + \frac{\varepsilon}{2}\left(\left|\frac{\partial u_\varepsilon}{\partial x_2}\right|^2 - \left|\frac{\partial u_\varepsilon}{\partial x_1}\right|^2\right)\right] dx_2, \quad k = 0, 1, 2, \cdots.$$

In the following, for simplicity we denote

$$\omega_\varepsilon := \frac{1}{\varepsilon}W(u_\varepsilon) + \frac{\varepsilon}{2}\left(\left|\frac{\partial u_\varepsilon}{\partial x_2}\right|^2 - \left|\frac{\partial u_\varepsilon}{\partial x_1}\right|^2\right).$$

Lemma 14.4. *In $(-1, 1)$, we have*

$$\frac{d}{dx_1} M_\varepsilon^0(x_1) = 0, \tag{14.2}$$

$$\frac{d^2}{dx_1^2} M_\varepsilon^1(x_1) = 0, \tag{14.3}$$

$$\frac{d^2}{dx_1^2}M_\varepsilon^2(x_1) = 2\int_{f_\varepsilon^-(x_1)}^{f_\varepsilon^+(x_1)} \frac{1}{\varepsilon}W(u_\varepsilon) - \frac{\varepsilon}{2}\left(\left|\frac{\partial u_\varepsilon}{\partial x_2}\right|^2 - \left|\frac{\partial u_\varepsilon}{\partial x_1}\right|^2\right) dx_2 \tag{14.4}$$
$$\geq 2\int_{f_\varepsilon^-(x_1)}^{f_\varepsilon^+(x_1)} \varepsilon\left|\frac{\partial u_\varepsilon}{\partial x_1}\right|^2 dx_2 \geq 0.$$

Proof. The proof is similar to the derivation of the Hamiltonian identity, by differentiating in x_1 and integrating by parts. By the free boundary condition, all of the boundary terms appearing in this procedure cancel with each other. The last two inequalities in (14.4) follow from the Modica inequality. □

Lemma 14.5. *There exists a constant E_ε satisfying*

$$\lim_{\varepsilon\to 0} E_\varepsilon = 0, \tag{14.5}$$

such that

$$M^0(x_1) \equiv 2\sigma_0 + E_\varepsilon \quad \text{in } (-1,1). \tag{14.6}$$

Moreover, we have

$$M_\varepsilon^1(x_1) = \left(\sigma_0 + \frac{E_\varepsilon}{2}\right)\left[f_\varepsilon^-(x_1) + f_\varepsilon^+(x_1)\right] + O(\varepsilon), \tag{14.7}$$

and

$$M_\varepsilon^2(x_1) = \left(\sigma_0 + \frac{E_\varepsilon}{2}\right)\left[f_\varepsilon^+(x_1)^2 + f_\varepsilon^-(x_1)^2\right] + O(\varepsilon). \tag{14.8}$$

Proof. (14.6) follows from the Hamiltonian identity and Hutchinson-Tonegawa theory.

Decompose $M_\varepsilon^1(x_1)$ as follows:

$$M_\varepsilon^1(x_1) = \int_{f_\varepsilon^-(x_1)}^{\tilde{f}_\varepsilon(x_1)} \left[x_2 - f_\varepsilon^-(x_1)\right]\omega_\varepsilon dx_2 + f_\varepsilon^-(x_1)\int_{f_\varepsilon^-(x_1)}^{\tilde{f}_\varepsilon(x_1)} \omega_\varepsilon dx_2 \tag{14.9}$$
$$+ \int_{\tilde{f}_\varepsilon(x_1)}^{f_\varepsilon^+(x_1)} \left[x_2 - f_\varepsilon^+(x_1)\right]\omega_\varepsilon dx_2 + f_\varepsilon^+(x_1)\int_{\tilde{f}_\varepsilon(x_1)}^{f_\varepsilon^+(x_1)} \omega_\varepsilon dx_2,$$

where $\tilde{f}_\varepsilon$ is chosen to satisfy

$$\sigma_0 + E_\varepsilon/2 = \int_{f_\varepsilon^-(x_1)}^{\tilde{f}_\varepsilon(x_1)} \omega_\varepsilon dx_2 = \int_{\tilde{f}_\varepsilon(x_1)}^{f_\varepsilon^+(x_1)} \omega_\varepsilon dx_2.$$

First, by the exponential decay (similar to Lemma 11.2)

$$1 - u_\varepsilon(x_1,x_2) \leq Ce^{-c\frac{\min\{x_2 - f_\varepsilon^-(x_1),\, f_\varepsilon^+(x_1) - x_2\}}{\varepsilon}}, \quad \text{in } \{u_\varepsilon > 0\}, \tag{14.10}$$

we have

$$\begin{aligned}\int_{f_\varepsilon^-(x_1)}^{\tilde{f}_\varepsilon(x_1)} \left[x_2 - f_\varepsilon^-(x_1)\right] \omega_\varepsilon dx_2 &\leq C\int_{f_\varepsilon^-(x_1)}^{\tilde{f}_\varepsilon(x_1)} \left[x_2 - f_\varepsilon^-(x_1)\right] e^{-c\frac{x_2 - f_\varepsilon^-(x_1)}{\varepsilon}} dx_2 \\ &\leq C\varepsilon \int_0^{\frac{\tilde{f}_\varepsilon(x_1) - f_\varepsilon^-(x_1)}{2\varepsilon}} x_2 e^{-cx_2} dx_2 \\ &\leq C\varepsilon.\end{aligned}$$

Similarly,

$$\int_{\tilde{f}_\varepsilon(x_1)}^{f_\varepsilon^+(x_1)} \left[x_2 - f_\varepsilon^+(x_1)\right] \omega_\varepsilon dx_2 = O(\varepsilon).$$

Substituting these two estimates into (14.9) we get (14.7).

The same argument also implies that

$$\int_{f_\varepsilon^-(x_1)}^{f_\varepsilon^+(x_1)} \left[x_2 - f_\varepsilon^-(x_1)\right] \left[x_2 - f_\varepsilon^+(x_1)\right] \omega_\varepsilon dx_2 = O(\varepsilon).$$

Expanding the right hand side gives

$$M_\varepsilon^2(x_1) - \left[f_\varepsilon^+(x_1) + f_\varepsilon^-(x_1)\right] M_\varepsilon^1(x_1) + f_\varepsilon^+(x_1) f_\varepsilon^-(x_1) M_\varepsilon^0(x_1) = O(\varepsilon).$$

Substituting (14.6) and (14.7) into this gives (14.8). $\square$

14.3 Decay estimate

The main tool to prove Theorem 14.1 is the following decay estimate, which is similar to Lemma 13.2. Once we have this decay estimate, the proof of Theorem 14.1 is exactly the same as the proof of Theorem 13.1.

Lemma 14.6. *For any $r \in (0, 1/2)$, either*

$$\max\left\{\left|f_\varepsilon^-(r)\right|, \left|f_\varepsilon^-(-r)\right|\right\} = o(\varepsilon),$$

or

$$\max\left\{\left|f_\varepsilon^-\left(\frac{r}{2}\right)\right|, \left|f_\varepsilon^-\left(-\frac{r}{2}\right)\right|\right\} \leq \frac{1}{8}\max\left\{\left|f_\varepsilon^-(r)\right|, \left|f_\varepsilon^-(-r)\right|\right\}.$$

Proof. Assume by the contrary, there exists a constant $\lambda > 0$, a sequence of functions u_ε and $f_\varepsilon^\pm$ satisfying the hypothesis **B1)-B6)**, and a sequence of $r_\varepsilon \in (0, 1/2)$ such that,

$$h_\varepsilon := \max\left\{\left|f_\varepsilon^-(r_\varepsilon)\right|, \left|f_\varepsilon^-(-r_\varepsilon)\right|\right\} \geq \lambda\varepsilon, \tag{14.11}$$

and

$$\max\left\{\left|f_\varepsilon^-\left(\frac{r_\varepsilon}{2}\right)\right|, \left|f_\varepsilon^-\left(-\frac{r_\varepsilon}{2}\right)\right|\right\} > \frac{1}{8}\max\left\{\left|f_\varepsilon^-(r_\varepsilon)\right|, \left|f_\varepsilon^-(-r_\varepsilon)\right|\right\}. \tag{14.12}$$

We will derive a contradiction from these two assumptions.

Since f_ε^- is a concave function satisfying $f_\varepsilon^-(0) = f_\varepsilon^{-\prime}(0) = 0$, $|f_\varepsilon^{-\prime\prime}| \leq 4$, and it converges to 0 uniformly, we have

$$\lim_{\varepsilon\to 0}\frac{h_\varepsilon}{r_\varepsilon} = 0. \tag{14.13}$$

Let

$$v_\varepsilon(x, y) := u_\varepsilon(r_\varepsilon x_1, h_\varepsilon x_2),$$

and

$$\tilde{f}_\varepsilon^\pm(x_1) := \frac{1}{h_\varepsilon} f_\varepsilon^\pm(r_\varepsilon x_1).$$

We have

1. v_ε is a solution of

$$\begin{cases} r_\varepsilon^{-2}\dfrac{\partial^2 v_\varepsilon}{\partial x_1^2} + h_\varepsilon^{-2}\dfrac{\partial^2 v_\varepsilon}{\partial x_2^2} = \varepsilon^{-2}W'(v_\varepsilon), & \text{in } \{v_\varepsilon > 0\} \cap \mathcal{C}_{r_\varepsilon^{-1}}(0), \\ \dfrac{\varepsilon h_\varepsilon}{2r_\varepsilon^2}\left|\dfrac{\partial v_\varepsilon}{\partial x_1}\right|^2 + \dfrac{\varepsilon}{2h_\varepsilon}\left|\dfrac{\partial v_\varepsilon}{\partial x_2}\right|^2 = \dfrac{h_\varepsilon}{\varepsilon}W(0), & \text{on } \partial\{v_\varepsilon > 0\}. \end{cases}$$

2. v_ε satisfies the Modica inequality

$$\frac{\varepsilon h_\varepsilon}{2r_\varepsilon^2}\left|\frac{\partial v_\varepsilon}{\partial x_1}\right|^2 + \frac{\varepsilon}{2h_\varepsilon}\left|\frac{\partial v_\varepsilon}{\partial x_2}\right|^2 \leq \frac{h_\varepsilon}{\varepsilon}W(v_\varepsilon), \quad \text{in } \{v_\varepsilon > 0\}. \tag{14.14}$$

3. $\tilde{f}_\varepsilon^-$ is a concave function satisfying $\tilde{f}_\varepsilon^-(0) = \tilde{f}_\varepsilon^{-\prime}(0) = 0$. $\tilde{f}_\varepsilon^+$ is a convex function satisfying $f_\varepsilon^+ > f_\varepsilon^-$. Moreover,

$$\{v_\varepsilon > 0\} \cap \mathcal{C}_{r_\varepsilon^{-1}}(0) = \{(x_1, x_2) : \tilde{f}_\varepsilon^-(x_1) < x_2 < \tilde{f}_\varepsilon^+(x_1), |x_1| < r_\varepsilon^{-1}\}.$$

4. We have

$$\max\left\{\left|\tilde{f}_\varepsilon^-(1)\right|, \left|\tilde{f}_\varepsilon^-(-1)\right|\right\} = 1, \tag{14.15}$$

and

$$\max\left\{\left|\tilde{f}_\varepsilon^-\left(\frac{1}{2}\right)\right|, \left|\tilde{f}_\varepsilon^-\left(-\frac{1}{2}\right)\right|\right\} \geq \frac{1}{8}. \tag{14.16}$$

By the last two conditions, we can assume $\tilde{f}_\varepsilon^-$ converges to a limit $\tilde{f}^-$ locally uniformly in $(-1,1)$, which is a concave function satisfying $\tilde{f}^-(0)=0$, $\tilde{f}\leq 0$,

$$\sup_{x_1\in(-1,1)}|\tilde{f}^-(x_1)|\leq 1,$$

and

$$\max\left\{\left|\tilde{f}^-\left(\frac{1}{2}\right)\right|,\left|\tilde{f}^-\left(-\frac{1}{2}\right)\right|\right\}\geq\frac{1}{8}. \tag{14.17}$$

In particular, $\tilde{f}^-$ is nontrivial.

Because $\tilde{f}_{\varepsilon,d}:=\tilde{f}_\varepsilon^+-\tilde{f}_\varepsilon^-$ is a positive convex function, we can assume it converges to a limit $\tilde{f}_d$, which is still a nonnegative convex function but could take the value $+\infty$.

Next the proof is divided into four cases.

Case 1. $\tilde{f}_d\equiv 0$ in $(-1,1)$.

By definition, this implies

$$f_\varepsilon^+(0)-f_\varepsilon^-(0)\ll h_\varepsilon.$$

Then by Lemma 14.2, as $\varepsilon\to 0$,

$$\frac{h_\varepsilon}{\varepsilon}\to+\infty. \tag{14.18}$$

Let

$$\tilde{\omega}_\varepsilon:=\frac{h_\varepsilon}{\varepsilon}W(v_\varepsilon)+\frac{\varepsilon}{2h_\varepsilon}\left|\frac{\partial v_\varepsilon}{\partial x_2}\right|^2-\frac{\varepsilon h_\varepsilon}{2r_\varepsilon^2}\left|\frac{\partial v_\varepsilon}{\partial x_1}\right|^2.$$

By a direct scaling, we get, for any $x_1\in(-1,1)$,

$$\widetilde{M}_\varepsilon^0(x_1):=\int_{\tilde{f}_\varepsilon^-(x_1)}^{\tilde{f}_\varepsilon^+(x_1)}\tilde{\omega}_\varepsilon(x_1,x_2)dx_2=M_\varepsilon^0(r_\varepsilon x_1)=2\sigma_0+E_\varepsilon, \tag{14.19}$$

and

$$\widetilde{M}_\varepsilon^1(x_1):=\int_{\tilde{f}_\varepsilon^-(x_1)}^{\tilde{f}_\varepsilon^+(x_1)}x_2\tilde{\omega}_\varepsilon(x_1,x_2)dx_2=\frac{1}{h_\varepsilon}M_\varepsilon^1(r_\varepsilon x_1), \tag{14.20}$$

which is a linear function of x_1.

A rescaling of (14.10) gives

$$1-v_\varepsilon(x_1,x_2)\leq Ce^{-c\frac{h_\varepsilon}{\varepsilon}\min\{x_2-\tilde{f}_\varepsilon^-(x_1),\tilde{f}_\varepsilon^+(x_1)-x_2\}},\quad\text{in }\{v_\varepsilon>0\}. \tag{14.21}$$

Then similar to Lemma 14.5, we have

$$\widetilde{M}_\varepsilon^1(x_1)=\left(\sigma_0+\frac{E_\varepsilon}{2}\right)\left[\tilde{f}_\varepsilon^+(x_1)+\tilde{f}_\varepsilon^-(x_1)\right]+O(\varepsilon/h_\varepsilon). \tag{14.22}$$

Because $\tilde{f}_\varepsilon^+ - \tilde{f}_\varepsilon^-$ converges to 0 uniformly on $[-3/4, 3/4]$, $\tilde{f}_\varepsilon^+$ converges to $\tilde{f}^-$ uniformly on $[-3/4, 3/4]$. By passing to the limit in (14.22) and using (14.5), (14.18) and (14.20), we see $\tilde{f}^-$ is a linear function. Since $\tilde{f}^- \le 0$ and $\tilde{f}^-(0) = 0$, this is only possbile if $\tilde{f}^- \equiv 0$ in $(-3/4, 3/4)$. This contradicts (14.17) and finishes the proof in this case.

Case 2. $\tilde{f}_d < +\infty$ in a closed subinterval I of $(-1, 1)$, and it equals $+\infty$ outside I.

For any $x_1 \in I$,

$$\lim_{\varepsilon \to 0} \frac{\tilde{f}_\varepsilon^+(x_1) - \tilde{f}_\varepsilon^-(x_1)}{h_\varepsilon} < +\infty.$$

Hence by Lemma 14.2, we still have (14.18), i.e. $h_\varepsilon \gg \varepsilon$.

As in Case 1, $\widetilde{M}_\varepsilon^1$ is a linear function in $(-1, 1)$ and (14.22) still holds. Because $\tilde{f}_\varepsilon^-$ is uniformly bounded in $(-1, 1)$, by our assumption $\tilde{f}_\varepsilon^+$ is also uniformly bounded on I. (Recall that $\tilde{f}_\varepsilon^+ - \tilde{f}_\varepsilon^-$ is a positive, convex function.) Combining this fact with (14.18) and (14.22) we deduce that $\tilde{M}_\varepsilon^1$ is uniformly bounded on I.

If I is not a single point, because $\widetilde{M}_\varepsilon^1$ is a linear function, $\widetilde{M}_\varepsilon^1$ is in fact uniformly bounded on any compact set of $(-1, 1)$. Hence by (14.22), (14.18) and the uniform bound on $\tilde{f}_\varepsilon^-$, we deduce that $\tilde{f}_\varepsilon^+$ is uniformly bounded on any compact set of $(-1, 1)$. This is a contradiction with our assumption that $\tilde{f}_\varepsilon^+ - \tilde{f}_\varepsilon^-$ goes to infinity in $(-1, 1) \setminus I$.

Next let us consider the case when I is a single point. Note that because $\tilde{f}_\varepsilon^-$ is uniformly bounded on $[-1, 1]$, $\tilde{f}_\varepsilon^+$ goes to $+\infty$ outside I. By (14.5), (14.22) and (14.18), $\widetilde{M}_\varepsilon^1$ also goes to $+\infty$ outside I. This is a contradiction with its linearity because $\widetilde{M}_\varepsilon^1$ is uniformly bounded on I.

Therefore in any case we arrive at a contradiction.

Case 3. $\tilde{f}_d < +\infty$ everywhere in $(-1, 1)$ and it does not equal 0 everywhere.

In Section 15 we will show that this leads to a contradiction.

Case 4. $\tilde{f}_d = +\infty$ everywhere in $(-1, 1)$.

In Section 16 we will show that this is also impossible. $\square$

15 Exclusion of Case 3

In this section we consider Case 3 in the proof of Lemma 14.6. That is, we assume (notations as in the proof of Lemma 14.6) $\tilde{f}_d < +\infty$ everywhere in $(-1, 1)$ and it does not equal 0 everywhere.

Consider

$$v_\epsilon(x_1, x_2) := u_\varepsilon(r_\varepsilon x_1, r_\varepsilon x_2),$$

where $\epsilon = \varepsilon / r_\varepsilon$.

1. v_ϵ is a solution of (12.2) with ε replaced by ϵ.

2. By **B2)** and the monotonicity formula (see [47]),

$$\int_{\mathcal{C}_1} \frac{\epsilon}{2}|\nabla v_\epsilon|^2 + \frac{1}{\epsilon} W(v_\epsilon)\chi_{\{v_\epsilon>0\}} \le C.$$

3. By **B3)** and a direct scaling, v_ϵ still satisfies the Modica inequality.

4. $\{v_\epsilon > 0\} = \{f_\epsilon^-(x_1) < x_2 < f_\epsilon^+(x_1)\}$, where

$$f_\epsilon^\pm(x_1) := \frac{1}{r_\varepsilon} f_\varepsilon^\pm(r_\varepsilon x_1).$$

5. By **B5)**,

$$\sup_{[-1,1]} |f_\epsilon^{\pm\prime\prime}(x_1)| \le C r_\varepsilon$$

are uniformly bounded.

6. By **B6)**,

$$\sup_{[-1,1]} |f_\epsilon^{\pm\prime}(x_1)| \le \sup_{[-1,1]} |f_\varepsilon^{\pm\prime}(x_1)|$$

converges to 0.

7. By (14.11) and (14.12), we have

$$\left|f_\epsilon^-(-1)\right| + \left|f_\epsilon^-(1)\right| = \frac{h_\varepsilon}{r_\varepsilon}$$

and

$$\left|f_\epsilon^-(-1/2)\right| + \left|f_\epsilon^-(1/2)\right| \ge \frac{1}{8}\frac{h_\varepsilon}{r_\varepsilon}.$$

8. $h_\varepsilon/r_\varepsilon \to 0$. There are two cases: if r_ε does not goes to 0, by **B6)** and (14.11), $h_\varepsilon \to 0$; if $r_\varepsilon \to 0$, by **B5)** and (14.11), $h_\varepsilon/r_\varepsilon \le C r_\varepsilon$ still converges to 0.

9. Since $f_\varepsilon^+(r_\varepsilon x_1)/h_\varepsilon$ are uniformly bounded on $[-1,1]$, there exists a constant M independent of ε such that

$$\sup_{x_1\in[-1,1]} \left|f_\epsilon^+(x_1)\right| \le M\frac{h_\epsilon}{r_\epsilon}.$$

10. Because $h_\varepsilon \gg \varepsilon$ (see Lemma 14.2),

$$\frac{h_\varepsilon}{r_\varepsilon} \gg \frac{\varepsilon}{r_\varepsilon} = \epsilon.$$

By abusing notations, we are in the following situation:

1. We have a sequence of solutions u_ε satisfying **B1)-B6)**, with $\varepsilon \to 0$.

2. There exists $h_\varepsilon \to 0$ satisfying $h_\varepsilon \gg \varepsilon$ so that

$$\lim_{\varepsilon} \frac{f_\varepsilon^\pm}{h_\varepsilon} = \tilde{f}^\pm$$

   uniformly on $[-1, 1]$.

3. $-\infty < \tilde{f}^- \le \tilde{f}^+ < +\infty$ on $[-1, 1]$.

We will show that $\tilde{f}^- \equiv 0$. As before, this leads to a contradiction.

For simplicity of presentation, we will take a rotation so that f_ε^+ and f_ε^- satisfy

$$f_\varepsilon^+ + f_\varepsilon^- = O(\varepsilon). \tag{15.1}$$

This is possible by using (14.7).

With this choice, the blow up limit satisfies $\tilde{f}^+ = -\tilde{f}^- \ge 0$.

15.1 An estimate on E_ε

Recall that (14.6) says, there exists a constant E_ε such that

$$\int_{f_\varepsilon^-(x_1)}^{f_\varepsilon^+(x_1)} \frac{1}{\varepsilon} W(u_\varepsilon) + \frac{\varepsilon}{2}\left(\left|\frac{\partial u_\varepsilon}{\partial x_2}\right|^2 - \left|\frac{\partial u_\varepsilon}{\partial x_1}\right|^2\right) = 2\sigma_0 + E_\varepsilon. \tag{15.2}$$

By Hutchinson-Tonegawa theory, $E_\varepsilon = o_\varepsilon(1)$.

In this subsection we will give an estimate of E_ε by modifying the calculation in Section 11 to the multiplicity 2 case.

For $(x_1, x_2) \in \{u_\varepsilon > 0\}$, let

$$g_*(x_1, x_2) := \max\left\{ g\left(\frac{x_2 - f_\varepsilon^-(x_1)}{\varepsilon}\right), g\left(\frac{f_\varepsilon^+(x_1) - x_2}{\varepsilon}\right)\right\}.$$

Note that g_* is continuous and it is smooth in $\{u_\varepsilon > 0\} \setminus \{x_2 = [f_\varepsilon^+(x_1) + f_\varepsilon^-(x_1)]/2\}$.

Define

$$w_\varepsilon(x_1, x_2) := g_*(x_1, x_2) - u_\varepsilon(x_1, x_2).$$

By definition, $w_\varepsilon = 0$ on $\{x_2 = f_\varepsilon^\pm(x_1)\}$. By the Modica inequality, $w_\varepsilon > 0$ in $\{f_\varepsilon^-(x_1) < x_2 < f_\varepsilon^+(x_1)\}$ (see Lemma 11.5).

In the following, we denote

$$f_\varepsilon(x_1) := \frac{f_\varepsilon^-(x_1) + f_\varepsilon^+(x_1)}{2} \tag{15.3}$$

and

$$g'_*(x_1,x_2) := \begin{cases} g'\left(\frac{x_2 - f_\varepsilon^-(x_1)}{\varepsilon}\right), & f_\varepsilon^-(x_1) < x_2 < f_\varepsilon(x_1), \\ g'\left(\frac{f_\varepsilon^+(x_1) - x_2}{\varepsilon}\right), & f_\varepsilon(x_1) < x_2 < f_\varepsilon^+(x_1), \end{cases}$$

$$g''_*(x_1,x_2) := \begin{cases} g''\left(\frac{x_2 - f_\varepsilon^-(x_1)}{\varepsilon}\right), & f_\varepsilon^-(x_1) < x_2 < f_\varepsilon(x_1), \\ g''\left(\frac{f_\varepsilon^+(x_1) - x_2}{\varepsilon}\right), & f_\varepsilon(x_1) < x_2 < f_\varepsilon^+(x_1). \end{cases}$$

Moreover,

$$\|w_\varepsilon(x_1)\|^2 := \int_{f_\varepsilon^-(x_1)}^{f_\varepsilon^+(x_1)} w_\varepsilon(x_1,x_2)^2 dx_2.$$

Similar to the calculation in [20, page 927], we have

$$\begin{aligned} & \int_{f_\varepsilon^-(x_1)}^{f_\varepsilon(x_1)} \frac{1}{\varepsilon} W(u_\varepsilon) + \frac{\varepsilon}{2}\left(\left|\frac{\partial u_\varepsilon}{\partial x_2}\right|^2 - \left|\frac{\partial u_\varepsilon}{\partial x_1}\right|^2\right) - \frac{\varepsilon}{2}|g'_*|^2 - \frac{1}{\varepsilon}W(g_*) \\ = & \int_{f_\varepsilon^-(x_1)}^{f_\varepsilon(x_1)} \frac{1}{\varepsilon}\left[W(u_\varepsilon) - W(g_*) - \frac{W'(u_\varepsilon) + W'(g_*)}{2}(u_\varepsilon - g_*)\right] \\ & + \frac{\varepsilon}{2}\int_{f_\varepsilon^-(x_1)}^{f_\varepsilon(x_1)} \left[(u_\varepsilon - g_*)\frac{\partial^2 u_\varepsilon}{\partial x_1^2} - \left|\frac{\partial u_\varepsilon}{\partial x_1}\right|^2\right] \\ + & \frac{\varepsilon}{2}\left[u_\varepsilon(x_1, f_\varepsilon(x_1)) - g_*(f_\varepsilon(x_1))\right]\left[\frac{\partial u_\varepsilon}{\partial x_2}(x_1, f_\varepsilon(x_1)) + \varepsilon^{-1} g'_*(x_1, f_\varepsilon(x_1))\right] \\ = & \frac{\varepsilon}{2}\int_{f_\varepsilon^-(x_1)}^{f_\varepsilon(x_1)} \left[(u_\varepsilon - g_*)\frac{\partial^2 u_\varepsilon}{\partial x_1^2} - \left|\frac{\partial u_\varepsilon}{\partial x_1}\right|^2\right] + o\left(\frac{\|w_\varepsilon(x_1)\|^2}{\varepsilon}\right) \\ + & \frac{\varepsilon}{2}\left[u_\varepsilon(x_1, f_\varepsilon(x_1)) - g_*(f_\varepsilon(x_1))\right]\left[\frac{\partial u_\varepsilon}{\partial x_2}(x_1, f_\varepsilon(x_1)) + \varepsilon^{-1} g'_*(x_1, f_\varepsilon(x_1))\right]. \end{aligned}$$

A similar one holds for the integral on $(f_\varepsilon(x_1), f_\varepsilon^+(x_1))$. Summing these two and using the Hamiltonian identity (15.2), we obtain

$$\begin{aligned} & \int_{f_\varepsilon^-(x_1)}^{f_\varepsilon^+(x_1)} \varepsilon\,(u_\varepsilon - g_*)\frac{\partial^2 u_\varepsilon}{\partial x_1^2} - \varepsilon\left|\frac{\partial u_\varepsilon}{\partial x_1}\right|^2 \\ = & \; 2E_\varepsilon + 4\int_{\frac{f_\varepsilon^+(x_1) - f_\varepsilon^-(x_1)}{2\varepsilon}}^{+\infty} |g'|^2 \end{aligned} \tag{15.4}$$

$$
\begin{aligned}
&- \quad 2\left[u_\varepsilon(x_1, f_\varepsilon(x_1)) - g_*(f_\varepsilon(x_1))\right] g'\left(\frac{f_\varepsilon^+(x_1) - f_\varepsilon^-(x_1)}{2\varepsilon}\right) + o\left(\frac{\|w_\varepsilon(x_1)\|^2}{\varepsilon}\right) \\
&\geq \quad \frac{2E_\varepsilon}{\varepsilon} + o\left(\frac{\|w_\varepsilon(x_1)\|^2}{\varepsilon}\right),
\end{aligned}
$$

where the last step follows from the positivity of $g_* - u_\varepsilon$.

Another estimate on E_ε is

Lemma 15.1. *For any* $x_1 \in (-1,1)$,

$$2E_\varepsilon \geq -\int_{f_\varepsilon^-(x_1)}^{f_\varepsilon^+(x_1)} \varepsilon \Big|\frac{\partial u_\varepsilon}{\partial x_1}(x_1, x_2)\Big|^2 dx_2.$$

This lemma is a direct consequence of the following estimate.

Lemma 15.2. *For any* $x_1 \in (-1,1)$,

$$\int_{f_\varepsilon^-(x_1)}^{f_\varepsilon^+(x_1)} \frac{1}{\varepsilon} W(u_\varepsilon) + \frac{\varepsilon}{2}\Big|\frac{\partial u_\varepsilon}{\partial x_2}\Big|^2 dx_2 \geq 2\sigma_0. \tag{15.5}$$

Proof. Extend $W(t)$ to $\{t < 0\}$ to be an even function. After this extension, W is still Lipschitz continuous on $[-1,1]$. (15.5) will follow from the claim:
Claim. For any $L > 0$,

$$e(L) := \min_{v \in H_0^1(0,L)} \int_0^L W(v) + \frac{1}{2}\Big|\frac{dv}{dt}\Big|^2 dt \geq 2\sigma_0. \tag{15.6}$$

Clearly this minima is attained by a function $v_L \in H_0^1(0,L)$. Replacing v_L by $|v_L|$ and $\min\{v_L, 1\}$, we must have $0 \leq v_L \leq 1$. Standard ordinary differential equation arguments imply that derivatives of v_L up to third order are uniformly bounded independent of L.

Take the test function

$$\tilde{v}_L(t) := \min\{g(t), g(L-t)\}.$$

By noting that g converges to 1 exponentially at infinity, we obtain

$$e(L) \leq 2\sigma_0 + Ce^{-cL}. \tag{15.7}$$

From this we deduce that

$$\max_{[0,L]} v_L \geq 1 - CL^{-\frac{1}{2}}. \tag{15.8}$$

Indeed, because $\inf_{[0,\gamma]} W > 0$, the length of $v_L^{-1}(0,\gamma)$ is bounded by a universal constant. On the other hand, in $v_L^{-1}(\gamma, 1)$,

$$W(v_L) \geq c\left(1 - v_L\right)^2.$$

Hence

$$\begin{aligned} C &\geq \int_{v_L^{-1}(\gamma,1)} (1-v_L)^2 \\ &\geq \int_{v_L^{-1}(\gamma,1)} \left(1-\max_{(0,L)} v_L\right)^2. \end{aligned}$$

From this we deduce (15.8).

Assume the maxima of v_L is attained at t_L.

Let

$$w_L(t) := \Phi(v_L(t)) = \int_0^{u_L(t)} \sqrt{2W(s)}ds.$$

Then

$$\begin{aligned} e_L &\geq \int_0^L \left|\frac{dw_L}{dt}\right| \\ &= \int_0^{t_L} \left|\frac{dw_L}{dt}\right| + \int_{t_L}^L \left|\frac{dw_L}{dt}\right| \\ &\geq 2w_L(t_L) \\ &= 2\int_0^{u_L(t_L)} \sqrt{2W(s)}ds. \end{aligned}$$

On the other hand,

$$\begin{aligned} \sigma_0 &= \int_0^{+\infty} \frac{1}{2}\left|g'(t)\right|^2 + W(g(t))dt \quad \text{(by definition)} \\ &= \int_0^{+\infty} \sqrt{2W(g(t))}g'(t)dt \quad \text{(by the first integral for } g) \\ &= \int_0^1 \sqrt{2W(s)}ds. \end{aligned}$$

By noting (15.8), we get

$$e_L \geq 2\sigma_0 + o_L(1). \tag{15.9}$$

To complete the proof, note that $e(L)$ is decreasing in L. Hence for any fixed L,

$$e_L \geq \lim_{L\to+\infty} e_L \geq 2\sigma_0,$$

where the last inequality follows from (15.9). □

15.2 An error estimate

Here we establish a technical estimate on the error between u_ε and a canonical one dimensional solution. This is the multiplicity 2 case of the result in Section 11.

First note that

Lemma 15.3. *There exists a constant $\mu > 0$ so that the following holds. For any $L > 0$, define*

$$g_L(t) := \max\{g(t+L), g(L-t)\}, \quad t \in [-L, L].$$

Then for any $\varphi \in H_0^1(-L, L)$,

$$\int_{-L}^{L} \left|\varphi'(t)\right|^2 + W''(g_L(t))\varphi(t)^2 dt \geq \mu \int_{-L}^{L} \varphi(t)^2 dt. \tag{15.10}$$

Proof. Because g is an increasing function, g_L is even in t. Let φ_L be the L^2 normalized first eigenfunction associated to (15.10). The corresponding eigenvalue is denoted by $\lambda(L)$. By the uniqueness of the first eigenfunction, φ_L is also even in t. Hence it satisfies $\varphi_L'(0) = 0$. Then $\varphi_L(-L+t)$, $t \in (0, L)$, is the minimizer of

$$\frac{\int_0^L |\varphi'(t)|^2 + W''(g(t))\varphi(t)^2 dt}{\int_0^L \varphi(t)^2 dt}$$

in the class of functions $\varphi \in H^1(0, L)$ satisfying $\varphi(0) = 0$. The remaining proof is similar to the derivation of (11.5) (via a contradiction argument). $\square$

Similar to (11.6), we can show that

$$\begin{aligned} \int_{f_\varepsilon^-(x_1)}^{f_\varepsilon^+(x_1)} (u_\varepsilon - g_*) \frac{\partial^2 u_\varepsilon}{\partial x_1^2} \quad &\geq \quad \frac{\mu + o(1)}{\varepsilon^2} \|w_\varepsilon(x_1)\|^2 \\ &+ \quad \frac{2}{\varepsilon} \left[u_\varepsilon(x_1, f_\varepsilon(x_1)) - g_*(f_\varepsilon(x_1))\right] g' \left(\frac{f_\varepsilon^+(x_1) - f_\varepsilon^-(x_1)}{2\varepsilon}\right). \end{aligned} \tag{15.11}$$

The last term comes from the boundary terms in the procedure of integration by parts.

Differentiating $\|w_\varepsilon\|^2$ twice in x_1 leads to (for the definition of f_ε see (15.3))

$$\begin{aligned} &\frac{1}{2}\frac{d^2}{dx_1^2}\|w_\varepsilon\|^2 \\ = \quad &\int_{f_\varepsilon^-(x_1)}^{f_\varepsilon(x_1)} \left|\frac{\partial u_\varepsilon}{\partial x_1} + \frac{1}{\varepsilon} g' \left(\frac{x_2 - f_\varepsilon^-(x_1)}{\varepsilon}\right) f_\varepsilon^{-\prime}(x_1)\right|^2 \end{aligned}$$

$$+ \int_{f_\varepsilon^-(x_1)}^{f_\varepsilon(x_1)} \left[\frac{\partial^2 u_\varepsilon}{\partial x_1^2} - \frac{f_\varepsilon^{-\prime}(x_1)^2}{\varepsilon^2} g'' \left(\frac{x_2 - f_\varepsilon^-(x_1)}{\varepsilon} \right) + \frac{f_\varepsilon^{-\prime\prime}(x_1)}{\varepsilon} g' \left(\frac{x_2 - f_\varepsilon^-(x_1)}{\varepsilon} \right) \right] (u_\varepsilon - g_*)$$

$$+ \int_{f_\varepsilon(x_1)}^{f_\varepsilon^+(x_1)} \left| \frac{\partial u_\varepsilon}{\partial x_1} - \frac{1}{\varepsilon} g' \left(\frac{f_\varepsilon^+(x_1) - x_2}{\varepsilon} \right) f_\varepsilon^{+\prime}(x_1) \right|^2$$

$$+ \int_{f_\varepsilon(x_1)}^{f_\varepsilon^+(x_1)} \left[\frac{\partial^2 u_\varepsilon}{\partial x_1^2} - \frac{f_\varepsilon^{+\prime}(x_1)^2}{\varepsilon^2} g' \left(\frac{f_\varepsilon^+(x_1) - x_2}{\varepsilon} \right) - \frac{f_\varepsilon^{+\prime\prime}(x_1)}{\varepsilon} g' \left(\frac{f_\varepsilon^+(x_1) - x_2}{\varepsilon} \right) \right] (u_\varepsilon - g_*)$$

$$+ \frac{1}{2\varepsilon} \left[u_\varepsilon(x_1, f_\varepsilon(x_1)) - g_*(f_\varepsilon(x_1)) \right] g' \left(\frac{f_\varepsilon^+(x_1) - f_\varepsilon^-(x_1)}{2\varepsilon} \right) \left[f_\varepsilon^{-\prime}(x_1) + f_\varepsilon^{+\prime}(x_1) \right]^2 .$$

A direct expansion gives

$$\begin{aligned}
&\frac{1}{2} \frac{d^2}{dx_1^2} \|w_\varepsilon\|^2 \\
= &\int_{f_\varepsilon^-(x_1)}^{f_\varepsilon^+(x_1)} \left| \frac{\partial u_\varepsilon}{\partial x_1} \right|^2 + \frac{\partial^2 u_\varepsilon}{\partial x_1^2} (u_\varepsilon - g_*) \\
+ &\int_{f_\varepsilon^-(x_1)}^{f_\varepsilon(x_1)} \frac{2}{\varepsilon} \frac{\partial u_\varepsilon}{\partial x_1} g_*' f_\varepsilon^{-\prime}(x_1) + \frac{\left| g_*' \right|^2 - g_*'' (u_\varepsilon - g_*)}{\varepsilon^2} f_\varepsilon^{-\prime}(x_1)^2 + \frac{g_*'}{\varepsilon} (u_\varepsilon - g_*) f_\varepsilon^{-\prime\prime}(x_1) \\
- &\int_{f_\varepsilon(x_1)}^{f_\varepsilon^+(x_1)} \frac{2}{\varepsilon} \frac{\partial u_\varepsilon}{\partial x_1} g_*' f_\varepsilon^{+\prime}(x_1) - \frac{\left| g_*' \right|^2 - g_*'' (u_\varepsilon - g_*)}{\varepsilon^2} f_\varepsilon^{+\prime}(x_1)^2 + \frac{g_*'}{\varepsilon} (u_\varepsilon - g_*) f_\varepsilon^{+\prime\prime}(x_1) \\
+ &\frac{1}{2\varepsilon} \left[u_\varepsilon(x_1, f_\varepsilon(x_1)) - g_*(f_\varepsilon(x_1)) \right] g' \left(\frac{f_\varepsilon^+(x_1) - f_\varepsilon^-(x_1)}{2\varepsilon} \right) \left[f_\varepsilon^{-\prime}(x_1) + f_\varepsilon^{+\prime}(x_1) \right]^2 .
\end{aligned}$$

Next, using (15.4) we have

$$\begin{aligned}
&\frac{1}{2} \frac{d^2}{dx_1^2} \|w_\varepsilon\|^2 \\
\geq &\int_{f_\varepsilon^-(x_1)}^{f_\varepsilon^+(x_1)} \frac{5}{3} \left| \frac{\partial u_\varepsilon}{\partial x_1} \right|^2 + \frac{1}{3} \frac{\partial^2 u_\varepsilon}{\partial x_1^2} (u_\varepsilon - g_*) \\
+ &\int_{f_\varepsilon^-(x_1)}^{f_\varepsilon(x_1)} \frac{2}{\varepsilon} \frac{\partial u_\varepsilon}{\partial x_1} g_*' f_\varepsilon^{-\prime}(x_1) + \frac{\left| g_*' \right|^2 - g_*'' (u_\varepsilon - g_*)}{\varepsilon^2} f_\varepsilon^{-\prime}(x_1)^2 \\
- &\int_{f_\varepsilon(x_1)}^{f_\varepsilon^+(x_1)} \frac{2}{\varepsilon} \frac{\partial u_\varepsilon}{\partial x_1} g_*' f_\varepsilon^{+\prime}(x_1) - \frac{\left| g_*' \right|^2 - g_*'' (u_\varepsilon - g_*)}{\varepsilon^2} f_\varepsilon^{+\prime}(x_1)^2 \\
&+ \frac{4E_\varepsilon}{3\varepsilon} + \frac{8}{3\varepsilon} \int_{\frac{f_\varepsilon^+(x_1) - f_\varepsilon^-(x_1)}{2\varepsilon}}^{+\infty} |g'|^2
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{1}{\varepsilon}\left[u_\varepsilon(x_1, f_\varepsilon(x_1)) - g_*(f_\varepsilon(x_1))\right] g'\left(\frac{f_\varepsilon(x_1) - f_\varepsilon^-(x_1)}{\varepsilon}\right)\left[-\frac{4}{3} + \frac{\left(f_\varepsilon^{-\prime}(x_1) + f_\varepsilon^{+\prime}(x_1)\right)^2}{2}\right] \\
&+ \quad o\left(\frac{\|w_\varepsilon\|^2}{\varepsilon^2}\right).
\end{aligned}
$$

Grouping this into squares and using (15.11) leads to

$$
\begin{aligned}
& \frac{1}{2}\frac{d^2}{dx_1^2}\|w_\varepsilon\|^2 \\
\geq \quad & \int_{f_\varepsilon^-(x_1)}^{f_\varepsilon(x_1)} \left|\sqrt{\frac{5}{3}}\frac{\partial u_\varepsilon}{\partial x_1} + \sqrt{\frac{3}{5}}\frac{1}{\varepsilon}g'\left(\frac{x_2 - f_\varepsilon^-(x_1)}{\varepsilon}\right) f_\varepsilon^{-\prime}(x_1)\right|^2 \\
+ \quad & \int_{f_\varepsilon(x_1)}^{f_\varepsilon^+(x_1)} \left|\sqrt{\frac{5}{3}}\frac{\partial u_\varepsilon}{\partial x_1} - \sqrt{\frac{3}{5}}\frac{1}{\varepsilon}g'\left(\frac{f_\varepsilon^+(x_1) - x_2}{\varepsilon}\right) f_\varepsilon^{+\prime}(x_1)\right|^2 \\
+ \quad & \frac{f_\varepsilon^{-\prime}(x_1)^2}{\varepsilon^2}\int_{f_\varepsilon^-(x_1)}^{f_\varepsilon(x_1)} \frac{2}{5}g'\left(\frac{x_2 - f_\varepsilon^-(x_1)}{\varepsilon}\right)^2 + g''\left(\frac{x_2 - f_\varepsilon^-(x_1)}{\varepsilon}\right) w_\varepsilon(x_1, x_2) \\
+ \quad & \frac{f_\varepsilon^{+\prime}(x_1)^2}{\varepsilon^2}\int_{f_\varepsilon(x_1)}^{f_\varepsilon^+(x_1)} \frac{2}{5}g'\left(\frac{f_\varepsilon^+(x_1) - x_2}{\varepsilon}\right)^2 + g''\left(\frac{f_\varepsilon^+(x_1) - x_2}{\varepsilon}\right) w_\varepsilon(x_1, x_2) \\
& + \frac{4E_\varepsilon}{3\varepsilon} + \frac{8}{3\varepsilon}\int_{\frac{f_\varepsilon^+(x_1) - f_\varepsilon^-(x_1)}{2\varepsilon}}^{+\infty} |g'|^2 \\
+ \quad & \frac{1}{\varepsilon}\left[u_\varepsilon(x_1, f_\varepsilon(x_1)) - g_*(f_\varepsilon(x_1))\right] g'\left(\frac{f_\varepsilon(x_1) - f_\varepsilon^-(x_1)}{\varepsilon}\right)\left[-\frac{4}{3} + \frac{\left(f_\varepsilon^{-\prime}(x_1) + f_\varepsilon^{+\prime}(x_1)\right)^2}{2}\right] \\
+ \quad & \frac{\mu + o(1)}{\varepsilon^2}\|w_\varepsilon\|^2 \\
\geq \quad & \frac{\mu}{2\varepsilon^2}\|w_\varepsilon\|^2 + \frac{c}{\varepsilon}\left[f_\varepsilon^{-\prime}(x_1)^2 + f_\varepsilon^{+\prime}(x_1)^2\right] - \frac{2E_\varepsilon^-}{\varepsilon}.
\end{aligned}
$$

Here $E_\varepsilon^- = \max\{0, -E_\varepsilon\}$. In the last step we have used the following facts.

- There exists a universal constant $c > 0$ such that

$$
\int_{f_\varepsilon^-(x_1)}^{f_\varepsilon(x_1)} \frac{2}{5}g'\left(\frac{x_2 - f_\varepsilon^-(x_1)}{\varepsilon}\right)^2 + g''\left(\frac{x_2 - f_\varepsilon^-(x_1)}{\varepsilon}\right) w_\varepsilon(x_1, x_2) \geq c\varepsilon,
$$

  and

$$
\int_{f_\varepsilon(x_1)}^{f_\varepsilon^+(x_1)} \frac{2}{5}g'\left(\frac{f_\varepsilon^+(x_1) - x_2}{\varepsilon}\right)^2 + g''\left(\frac{f_\varepsilon^+(x_1) - x_2}{\varepsilon}\right) w_\varepsilon(x_1, x_2) \geq c\varepsilon.
$$

- Because $g' > 0$,

$$-w_\varepsilon(x_1, f_\varepsilon(x_1)) = u_\varepsilon(x_1, f_\varepsilon(x_1)) - g_*(f_\varepsilon(x_1)) < 0$$

  and

$$-\frac{4}{3} + \frac{\left(f_\varepsilon^{-\prime}(x_1) + f_\varepsilon^{+\prime}(x_1)\right)^2}{2} < 0,$$

  the boundary term

$$-\frac{1}{\varepsilon} w_\varepsilon(x_1, f_\varepsilon(x_1)) g' \left(\frac{f_\varepsilon(x_1) - f_\varepsilon^-(x_1)}{\varepsilon}\right) \left[-\frac{2}{3} + \frac{\left(f_\varepsilon^{-\prime}(x_1) + f_\varepsilon^{+\prime}(x_1)\right)^2}{2}\right] > 0.$$

From this differential inequality we deduce that,

$$\|w_\varepsilon\|^2 \le C E_\varepsilon^- \varepsilon + C e^{-c\varepsilon^{-1}}. \tag{15.12}$$

15.3 A convex function

Similar to [42], by the stationary condition and the Modica inequality, we can define a convex function V_ε with

$$\nabla^2 V_\varepsilon = \begin{bmatrix} \frac{1}{\varepsilon} W(u_\varepsilon) - \frac{\varepsilon}{2}\left(\frac{\partial u_\varepsilon}{\partial x_2}\right)^2 + \frac{\varepsilon}{2}\left(\frac{\partial u_\varepsilon}{\partial x_1}\right)^2 & \varepsilon \frac{\partial u_\varepsilon}{\partial x_1}\frac{\partial u_\varepsilon}{\partial x_2} \\ \varepsilon \frac{\partial u_\varepsilon}{\partial x_1}\frac{\partial u_\varepsilon}{\partial x_2} & \frac{1}{\varepsilon} W(u_\varepsilon) + \frac{\varepsilon}{2}\left(\frac{\partial u_\varepsilon}{\partial x_2}\right)^2 - \frac{\varepsilon}{2}\left(\frac{\partial u_\varepsilon}{\partial x_1}\right)^2 \end{bmatrix}.$$

After subtracting a linear function, we can assume that $V_\varepsilon(0) = 0$ and $\nabla V_\varepsilon(0) = 0$. Hence by the convexity, $V_\varepsilon \ge 0$.

By direct calculation using the free boundary condition, we can show that

$$\frac{d}{dx_1} \nabla V_\varepsilon(x_1, f_\varepsilon^\pm(x_1)) = 0.$$

Thus $\nabla V_\varepsilon = 0$ on $\{x_2 = f_\varepsilon^-(x_1)\}$ and it is a constant vector on $\{x_2 = f_\varepsilon^+(x_1)\}$. Indeed, by (14.6),

$$\frac{\partial V_\varepsilon}{\partial x_2}(x_1, f_\varepsilon^+(x_1)) = \int_{f_\varepsilon^-(x_1)}^{f_\varepsilon^+(x_1)} \frac{1}{\varepsilon} W(u_\varepsilon) + \frac{\varepsilon}{2}\left(\frac{\partial u_\varepsilon}{\partial x_2}\right)^2 - \frac{\varepsilon}{2}\left(\frac{\partial u_\varepsilon}{\partial x_1}\right)^2 dx_2 = 2\sigma_0 + E_\varepsilon. \tag{15.13}$$

We will extend V_ε to be 0 in $\{x_2 < f_\varepsilon^-(x_1)\}$ and to be a linear function in $\{x_2 > f_\varepsilon^+(x_1)\}$. This allows us to view V_ε as a convex function defined in the whole $\mathcal{C}_1$.

By (15.13) and the convexity of V_ε, we deduce that, for any (x_1, x_2) with $f_\varepsilon^-(x_1) < x_2 < f_\varepsilon^+(x_2)$,

$$0 \leq \frac{\partial V_\varepsilon}{\partial x_2}(x_1, x_2) \leq 2\sigma_0 + E_\varepsilon.$$

Hence in this domain we have

$$V_\varepsilon(x_1, x_2) \leq (2\sigma_0 + E_\varepsilon)\left[x_2 - f_\varepsilon^-(x_1)\right]. \tag{15.14}$$

Define

$$\widetilde{V}_\varepsilon(x_1, x_2) := \frac{1}{h_\varepsilon} V_\varepsilon(x_1, h_\varepsilon x_2). \tag{15.15}$$

It is still a convex function. Moreover, $\widetilde{V}_\varepsilon = 0$ and $\nabla \widetilde{V}_\varepsilon = 0$ on $\{x_2 = \tilde{f}_\varepsilon^-(x_1)\}$. Rescaling (15.14) gives

$$\widetilde{V}_\varepsilon(x_1, x_2) \leq (2\sigma_0 + E_\varepsilon)\left[x_2 - \tilde{f}_\varepsilon^-(x_1)\right], \quad \text{for } \tilde{f}_\varepsilon^-(x_1) < x_2 < \tilde{f}_\varepsilon^+(x_1). \tag{15.16}$$

Hence $\widetilde{V}_\varepsilon$ are uniformly bounded. Then by the convexity, we can assume that $\widetilde{V}_\varepsilon$ converges uniformly to a convex function $\widetilde{V}$ on any compact set of $\mathcal{C}_1$. Note that $\widetilde{V} = 0$ in $\{x_2 \leq \tilde{f}^-(x_1)\}$ and it is linear in $\{x_2 \geq \tilde{f}^+(x_1)\}$.

We have

$$\nabla^2 \widetilde{V}_\varepsilon = \begin{bmatrix} \frac{1}{h_\varepsilon}\left[\frac{1}{\varepsilon}W(u_\varepsilon) - \frac{\varepsilon}{2}\left(\frac{\partial u_\varepsilon}{\partial x_2}\right)^2 + \frac{\varepsilon}{2}\left(\frac{\partial u_\varepsilon}{\partial x_1}\right)^2\right] & \varepsilon \frac{\partial u_\varepsilon}{\partial x_1}\frac{\partial u_\varepsilon}{\partial x_2} \\ \varepsilon \frac{\partial u_\varepsilon}{\partial x_1}\frac{\partial u_\varepsilon}{\partial x_2} & h_\varepsilon\left[\frac{1}{\varepsilon}W(u_\varepsilon) + \frac{\varepsilon}{2}\left(\frac{\partial u_\varepsilon}{\partial x_2}\right)^2 - \frac{\varepsilon}{2}\left(\frac{\partial u_\varepsilon}{\partial x_1}\right)^2\right] \end{bmatrix},$$

where the left hand side is evaluated at (x_1, x_2) while the right hand side is evaluated at $(x_1, h_\varepsilon x_2)$.

Take an interval I where $\tilde{f}^- < \tilde{f}^+$. For any $\delta > 0$, in $\{f_\varepsilon^-(x_1) + \delta h_\varepsilon < x_2 < f_\varepsilon^+(x_1) - \delta h_\varepsilon, x_1 \in I\}$, by (14.21),

$$\frac{1}{\varepsilon}W(u_\varepsilon) + \frac{\varepsilon}{2}\left(\frac{\partial u_\varepsilon}{\partial x_2}\right)^2 - \frac{\varepsilon}{2}\left(\frac{\partial u_\varepsilon}{\partial x_1}\right)^2 \leq \frac{C}{\varepsilon} e^{-\frac{ch_\varepsilon}{\varepsilon}}.$$

Thus in $\{\tilde{f}_\varepsilon^-(x_1) + \delta < x_2 < \tilde{f}_\varepsilon^+(x_1) - \delta, x_1 \in I\}$,

$$\frac{\partial^2 \widetilde{V}_\varepsilon}{\partial x_2^2} \leq \frac{C h_\varepsilon}{\varepsilon} e^{-\frac{ch_\varepsilon}{\varepsilon}} \to 0,$$

because $h_\varepsilon / \varepsilon \to +\infty$.

Passing to the limit and noting that δ can be arbitrarily small, we see $\frac{\partial^2 \widetilde{V}}{\partial x_2^2} = 0$ in $\{\tilde{f}^-(x_1) < x_2 < \tilde{f}^+(x_1), x_1 \in I\}$. By the convexity of $\widetilde{V}$, we also have $\frac{\partial^2 \widetilde{V}}{\partial x_1 \partial x_2} = 0$ in $\{\tilde{f}^-(x_1) < x_2 < \tilde{f}^+(x_1), x_1 \in I\}$.

Hence in the open set $\{(x_1, x_2) : x_1 \in I, \tilde{f}^-(x_1) < x_2 < \tilde{f}^+(x_1)\}$, there exists a function $\widetilde{W}$ defined on I such that

$$\widetilde{V}(x_1, x_2) = \sigma_0 \left[x_2 - \widetilde{W}(x_1)\right]. \tag{15.17}$$

Here we have used the fact that, for any $\delta > 0$,

$$\lim_{\varepsilon \to 0} \int_{\tilde{f}_\varepsilon^-(x_1)}^{\tilde{f}_\varepsilon^-(x_1)+\delta} \frac{\partial^2 \widetilde{V}_\varepsilon}{\partial x_2^2}(x_1, x_2) dx_2 = \sigma_0.$$

This follows from Lemma 14.3 after a scaling.

Because $\widetilde{V}$ is continuous, $\widetilde{W}$ is continuous. Recall that $\widetilde{V} = 0$ on $\{x_2 = \tilde{f}^-(x_1)\}$. By continuity, $\widetilde{W} = \tilde{f}^-$.

Recall that V_ε is a linear function above $\{x_2 = f_\varepsilon^+(x_1)\}$. By (15.13) and passing to the limit, we see in $\{x_2 > \tilde{f}^+(x_1)\}$,

$$\widetilde{V}(x_1, x_2) = 2\sigma_0 x_2 + \beta x_1,$$

for some constant β. Using convexity from both sides of $\{x_2 = \tilde{f}^+(x_1)\}$ and the fact that $\tilde{f}^+ = -\tilde{f}^-$, we deduce that $\beta = 0$.

In conclusion,

$$\widetilde{V}(x_1, x_2) = \begin{cases} 0, & \text{in } \{x_2 \leq \tilde{f}^-(x_1)\}, \\ \sigma_0 \left[x_2 - \tilde{f}^-(x_1)\right], & \text{in } \{\tilde{f}^-(x_1) < x_2 < \tilde{f}^+(x_1)\}, \\ 2\sigma_0 x_2, & \text{in } \{x_2 > \tilde{f}^+(x_1)\}. \end{cases}$$

15.4 An estimate on $\int \varepsilon \left|\frac{\partial^2 u_\varepsilon}{\partial x_1^2}\right|^2$

Recall that there exists a constant M independent of ε, such that

$$\sup_{x_1 \in (-1,1)} \left(\left|f_\varepsilon^+(x_1)\right| + \left|f_\varepsilon^-(x_1)\right|\right) \leq M h_\varepsilon.$$

Then by the definition of M_ε^2 (noting that for $(x_1, x_2) \in \{u_\varepsilon > 0\}$, $f_\varepsilon^-(x_1) < x_2 < f_\varepsilon^+(x_1)$),

$$\sup_{x_1 \in (-1,1)} M_\varepsilon^2(x_1) \leq C M^2 h_\varepsilon^2.$$

Hence using (14.4) we obtain, for any $\delta > 0$,

$$\int_{-1+\delta}^{1-\delta} \int_{f_\varepsilon^-(x_1)}^{f_\varepsilon^+(x_1)} \varepsilon \Big| \frac{\partial u_\varepsilon}{\partial x_1}(x_1, x_2) \Big|^2 dx_2 dx_1 \le C(\delta) M^2 h_\varepsilon^2. \tag{15.18}$$

The next result gives an estimate on the second order derivative $\frac{\partial^2 u_\varepsilon}{\partial x_1^2}$.

Lemma 15.4.

$$\int_{-5/6}^{5/6} \int_{f_\varepsilon^-(x_1)}^{f_\varepsilon^+(x_1)} \varepsilon \Big| \frac{\partial^2 u_\varepsilon}{\partial x_1^2}(x_1, x_2) \Big|^2 dx_2 dx_1 = o\left(\frac{h_\varepsilon^2}{\varepsilon^2} \right). \tag{15.19}$$

Proof. **Step 1.** For $x \in \{u_\varepsilon > 0\}$, let $d_\varepsilon(x)$ be the distance to $\partial\{u_\varepsilon > 0\}$. By Lemma 14.2, there exists a constant $\bar{L}$ independent of ε such that, in $\{d(x) > \bar{L}\varepsilon\}$, $u_\varepsilon \ge \gamma$. Then a direct calculation shows that in this set

$$\varepsilon \Delta \Big| \frac{\partial u_\varepsilon}{\partial x_1} \Big|^2 \ge 2\varepsilon \left(\Big| \frac{\partial^2 u_\varepsilon}{\partial x_1^2} \Big|^2 + \Big| \frac{\partial^2 u_\varepsilon}{\partial x_1 \partial x_2} \Big|^2 \right). \tag{15.20}$$

Take a smooth function η defined on $\mathbb{R}$ satisfying $\eta \equiv 1$ on $[2, +\infty)$, $\eta \equiv 0$ on $(-\infty, 1]$ and $|\eta'| \le 2$. Take another cut-off function $\zeta \in C_0^\infty(-6/7, 6/7)$ with $\zeta \equiv 1$ in $(-5/6, 5/6)$ and $|\zeta'| \le 100$. For any $L > \bar{L}$, multiplying (15.20) by $\eta\left(\frac{d_\varepsilon(x)}{L\varepsilon}\right)^2 \zeta(x_1)^2$ and integrating by parts leads to

$$\begin{aligned}
& \int_{\mathcal{C}_1 \cap \{u_\varepsilon > 0\}} \varepsilon \left(\Big| \frac{\partial^2 u_\varepsilon}{\partial x_1^2} \Big|^2 + \Big| \frac{\partial^2 u_\varepsilon}{\partial x_1 \partial x_2} \Big|^2 \right) \eta\left(\frac{d(x)}{L\varepsilon} \right)^2 \zeta(x_1)^2 \\
\le \ & -C \int_{\mathcal{C}_1 \cap \{u_\varepsilon > 0\}} \varepsilon \nabla \Big| \frac{\partial u_\varepsilon}{\partial x_1} \Big|^2 \nabla \left[\eta\left(\frac{d(x)}{L\varepsilon} \right)^2 \zeta(x_1)^2 \right].
\end{aligned}$$

Using Cauchy-Schwarz inequality we obtain

$$\begin{aligned}
& \int_{\mathcal{C}_1 \cap \{u_\varepsilon > 0\}} \varepsilon \left(\Big| \frac{\partial^2 u_\varepsilon}{\partial x_1^2} \Big|^2 + \Big| \frac{\partial^2 u_\varepsilon}{\partial x_1 \partial x_2} \Big|^2 \right) \eta\left(\frac{d(x)}{L\varepsilon} \right)^2 \zeta(x_1)^2 \\
\le \ & C \int_{\mathcal{C}_1 \cap \{u_\varepsilon > 0\}} \varepsilon \nabla \Big| \frac{\partial u_\varepsilon}{\partial x_1} \Big|^2 \left[\eta\left(\frac{d(x)}{L\varepsilon} \right)^2 \zeta'(x_1)^2 + \frac{1}{L^2\varepsilon^2} \eta'\left(\frac{d(x)}{L\varepsilon} \right)^2 \zeta(x_1)^2 \right] \\
\le \ & C h_\varepsilon^2 + C L^{-2} e^{-cL} \frac{h_\varepsilon^2}{\varepsilon^2},
\end{aligned}$$

where in the last step we have used (15.18) and [46, Lemma B.4].

Step 2. By Lemma 14.3, for any $L > 0$ fixed, if ε is small enough, $\frac{\partial u_\varepsilon}{\partial x_2} \neq 0$ in $\{0 < d_\varepsilon(x) < L\varepsilon\}$. Hence

$$\psi_\varepsilon := \frac{\partial u_\varepsilon}{\partial x_1} \Big/ \frac{\partial u_\varepsilon}{\partial x_2}$$

is well defined in this set.

As in Lemma 8.1, ψ_ε satisfies

$$\begin{cases} \operatorname{div}\left(\Big|\dfrac{\partial u_\varepsilon}{\partial x_2}\Big|^2 \nabla \psi_\varepsilon\right) = 0, & \text{in } \{0 < d_\varepsilon(x) < L\varepsilon\}, \\ \dfrac{\partial \psi_\varepsilon}{\partial \nu_\varepsilon} = 0, \quad \text{on } \partial\{u_\varepsilon > 0\}, \end{cases}$$

where ν_ε is the normal vector of $\partial\{u_\varepsilon > 0\}$.

Multiplying this equation by $\psi_\varepsilon \left[1 - \eta\left(\frac{d_\varepsilon}{L\varepsilon}\right)\right]^2 \zeta(x_1)^2$ and integrating by parts gives

$$\begin{aligned} & \int_{\mathcal{C}_1 \cap \{u_\varepsilon > 0\}} \varepsilon \Big|\frac{\partial u_\varepsilon}{\partial x_2}\Big|^2 |\nabla \psi_\varepsilon|^2 \left[1 - \eta\left(\frac{d(x)}{L\varepsilon}\right)\right]^2 \zeta(x_1)^2 \\ = & -\int_{\mathcal{C}_1 \cap \{u_\varepsilon > 0\}} \varepsilon \Big|\frac{\partial u_\varepsilon}{\partial x_2}\Big|^2 \psi_\varepsilon \nabla \psi_\varepsilon \cdot \nabla \left[1 - \eta\left(\frac{d(x)}{L\varepsilon}\right)\right]^2 \zeta(x_1)^2. \end{aligned}$$

Using the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} & \int_{\mathcal{C}_{5/6} \cap \{0 < d_\varepsilon < L\varepsilon\}} \varepsilon \Big|\frac{\partial u_\varepsilon}{\partial x_2}\Big|^2 |\nabla \psi_\varepsilon|^2 \\ \leq & \ C \int_{\mathcal{C}_1 \cap \{u_\varepsilon > 0\}} \varepsilon \Big|\frac{\partial u_\varepsilon}{\partial x_1}\Big|^2 \left[\zeta'(x_1)^2 + \frac{1}{L^2\varepsilon^2} \eta'\left(\frac{d(x)}{L\varepsilon}\right)^2\right] \\ \leq & \ C h_\varepsilon^2 + C L^2 e^{-cL} \frac{h_\varepsilon^2}{\varepsilon^2}, \end{aligned} \tag{15.21}$$

where (15.18) and [46, Lemma B.4] was also used in the last step.

Direct differentiation shows that

$$\varepsilon \Big|\frac{\partial u_\varepsilon}{\partial x_2}\Big|^2 |\nabla \psi_\varepsilon|^2 = \varepsilon \Big|\frac{\partial^2 u_\varepsilon}{\partial x_1^2} - \frac{\partial^2 u_\varepsilon}{\partial x_1 \partial x_2} \psi_\varepsilon\Big|^2 + \varepsilon \Big|\frac{\partial^2 u_\varepsilon}{\partial x_1 \partial x_2} - \frac{\partial^2 u_\varepsilon}{\partial x_2^2} \psi_\varepsilon\Big|^2. \tag{15.22}$$

We will estimate these four terms one by one. In the following, L will be fixed so that it is independent of ε.

First, because $\frac{\partial u_\varepsilon}{\partial x_2} > 0$ (or < 0) in $\{0 < d_\varepsilon < L\varepsilon\}$ and it satisfies the linearized equation (7.1) (scaled by ε), by standard interior gradient estimate and Harnack inequality, we see for any $x \in \{0 < d_\varepsilon < L\varepsilon\}$,

$$\Big|\frac{\partial^2 u_\varepsilon}{\partial x_2^2}(x)\Big| \leq \frac{C}{\varepsilon} \sup_{B_\varepsilon(x) \cap \{u_\varepsilon > 0\}} \Big|\frac{\partial u_\varepsilon}{\partial x_2}\Big| \leq C \Big|\frac{\partial u_\varepsilon}{\partial x_2}(x)\Big|.$$

It should be emphasized that the constant C in the above estimate is universal. It depends only on the double well potential W and is independent of L. By this estimate we get

$$\int_{\mathcal{C}_{5/6}\cap\{0<d_\varepsilon<L\varepsilon\}}\varepsilon\Big|\frac{\partial^2 u_\varepsilon}{\partial x_2^2}\psi_\varepsilon\Big|^2\le\frac{C}{\varepsilon^2}\int_{\mathcal{C}_{5/6}\cap\{0<d_\varepsilon<L\varepsilon\}}\varepsilon\Big|\frac{\partial u_\varepsilon}{\partial x_1}\Big|^2\le C\frac{h_\varepsilon^2}{\varepsilon^2}. \tag{15.23}$$

Combining this with (15.21) and (15.22) we get

$$\int_{\mathcal{C}_{5/6}\cap\{0<d_\varepsilon<L\varepsilon\}}\varepsilon\Big|\frac{\partial^2 u_\varepsilon}{\partial x_1\partial x_2}\Big|^2\le C\frac{h_\varepsilon^2}{\varepsilon^2}. \tag{15.24}$$

Recall that in $\{0<d_\varepsilon<L\varepsilon\}$, $|\psi_\varepsilon|\to 0$ uniformly (see Lemma 14.3). Thus

$$\int_{\mathcal{C}_{5/6}\cap\{0<d_\varepsilon<L\varepsilon\}}\varepsilon\Big|\frac{\partial^2 u_\varepsilon}{\partial x_1\partial x_2}\psi_\varepsilon\Big|^2=o\left(\frac{h_\varepsilon^2}{\varepsilon^2}\right). \tag{15.25}$$

Then combining (15.21), (15.22) and (15.24) we obtain

$$\int_{\mathcal{C}_{5/6}\cap\{0<d_\varepsilon<L\varepsilon\}}\varepsilon\Big|\frac{\partial^2 u_\varepsilon}{\partial x_1^2}\Big|^2=o\left(\frac{h_\varepsilon^2}{\varepsilon^2}\right). \tag{15.26}$$

Step 3. For any $\delta>0$, choose an L large so that $CL^2e^{-cL}<\delta$. Combining (15.21) and (15.25) we see

$$\lim_{\varepsilon\to 0}\frac{h_\varepsilon^2}{\varepsilon^2}\int_{\mathcal{C}_{5/6}}\varepsilon\Big|\frac{\partial^2 u_\varepsilon}{\partial x_1^2}\Big|^2\le\delta.$$

Since δ can be arbitrarily small, the proof of this lemma is complete. □

15.5 The blow up limit is piecewise linear

Combining Lemma 15.1 and (15.18), we get

$$E_\varepsilon\ge -Ch_\varepsilon^2. \tag{15.27}$$

Combining Lemma 15.4 with (15.27), (15.4) and (15.12), we obtain

$$-\int_{f_\varepsilon^-(x_1)}^{f_\varepsilon^+(x_1)}\varepsilon\Big|\frac{\partial u_\varepsilon}{\partial x_1}(x_1,x_2)\Big|^2dx_2\le 2E_\varepsilon+o(h_\varepsilon^2),\quad\forall x_1\in(-5/6,5/6).$$

This, in combination with Lemma 15.1, implies that

$$-\int_{f_\varepsilon^-(x_1)}^{f_\varepsilon^+(x_1)}\varepsilon\Big|\frac{\partial u_\varepsilon}{\partial x_1}(x_1,x_2)\Big|^2dx_2=2E_\varepsilon+o(h_\varepsilon^2),\quad\forall x_1\in(-5/6,5/6). \tag{15.28}$$

A consequence of (15.28) is that, for all ε small, $0 > E_\varepsilon \geq -Ch_\varepsilon^2$. Therefore, after subtracting a subsequence, we may assume there exists a constant $\tau \geq 0$ such that

$$\lim_{\varepsilon\to 0}\frac{E_\varepsilon}{h_\varepsilon^2} = -\tau^2\sigma_0. \tag{15.29}$$

Proposition 15.5. *In* $(-5/6, 5/6)$, $\tilde{f}^{+\prime}(x_1)^2 \equiv \tilde{f}^{-\prime}(x_1)^2 \equiv \tau^2$.

Proof. **Step 1.** For any $\varphi \in C_0^\infty(-5/6, 5/6)$,

$$\liminf_{\varepsilon\to 0}\frac{1}{h_\varepsilon^2}\int_{-5/6}^{5/6}\int_{f_\varepsilon^-(x_1)}^{f_\varepsilon^+(x_1)}\varepsilon\left(\frac{\partial u_\varepsilon}{\partial x_1}\right)^2\varphi(x_1) \geq \sigma_0\int_{-5/6}^{5/6}\left[\tilde{f}^{+\prime}(x_1)^2 + \tilde{f}^{-\prime}(x_1)^2\right]\varphi(x_1)dx_1. \tag{15.30}$$

This can be proven by noting that for any fixed $b > 0$, the level set $\{u_\varepsilon = t\}$, $t \in [0, 1-b]$, can be represented by the graph $\{x_2 = f_\varepsilon^\pm(x_1, t)\}$, where

$$f_\varepsilon^-(x_1, t) - f_\varepsilon^-(x_1) \leq C(b)\varepsilon, \quad f_\varepsilon^+(x_1) - f_\varepsilon^+(x_1, t) \leq C(b)\varepsilon, \quad \forall t \in [0, 1-b], \tag{15.31}$$

see Lemma 14.3.

Using this representation we can transform the integral in the right hand side of (15.30) (only for the part on $\{0 < u_\varepsilon < 1-b\}$) to an integral involving $\frac{\partial f_\varepsilon^\pm(x_1,t)}{\partial x_1}$. The claim then follows from standard lower semi-continuity of H^1 norm and the fact that $f_\varepsilon^\pm(x_1, t)/h_\varepsilon$ converges to $\tilde{f}^\pm$ in $L^2(-5/6, 5/6)$. (Recall that this is known for $t = 0$, while for $t \in (0, 1-b)$ this follows from (15.31) and the fact that $h_\varepsilon \gg \varepsilon$.) For more details, see [46, Section 8].

Step 2. Recall that the convex function $\widetilde{V}_\varepsilon$, introduced in Subsection 15.3, converges uniformly to $\widetilde{V}$ in $\mathcal{C}_{5/6}$. Then by the convexity, $\frac{\partial^2 \widetilde{V}_\varepsilon}{\partial x_1^2}$ converges to $\frac{\partial^2 \widetilde{V}}{\partial x_1^2}$ as Radon measures.

By the main result in Subsection 15.3, for any $\varphi \in C_0^\infty(-5/6, 5/6)$ and $\psi \in C_0^\infty(\mathbb{R})$,

$$\begin{aligned}
\int_{\mathcal{C}_{5/6}}\frac{\partial^2\widetilde{V}}{\partial x_1^2}\varphi(x_1)\psi(x_2) &= \sigma_0\int_{-5/6}^{5/6}\tilde{f}^{+\prime}(x_1)^2\varphi(x_1)\psi(\tilde{f}^+(x_1))dx_1 \\
&+ \sigma_0\int_{-5/6}^{5/6}\tilde{f}^{-\prime}(x_1)^2\varphi(x_1)\psi(\tilde{f}^-(x_1))dx_1 \\
&+ \sigma_0\int_{-5/6}^{5/6}\int_{\tilde{f}^-(x_1)}^{\tilde{f}^+(x_1)}\tilde{f}^{+\prime\prime}(x_1)\varphi(x_1)\psi(x_2)dx_2dx_1.
\end{aligned}$$

Here $\tilde{f}^{+\prime\prime}$ is understood as a Radon measure.

By the convergence of $\frac{\partial^2 \widetilde{V}_\varepsilon}{\partial x_1^2}$, this implies that

$$\lim_{\varepsilon\to 0}\frac{1}{h_\varepsilon^2}\int_{\mathcal{C}_{5/6}}\left[\frac{1}{\varepsilon}W(u_\varepsilon) - \frac{\varepsilon}{2}\left|\frac{\partial u_\varepsilon}{\partial x_2}\right|^2 + \frac{\partial u_\varepsilon}{\partial x_1}\right|^2\right]\varphi(x_1)\psi(h_\varepsilon x_2)$$

$$
\begin{aligned}
&= \sigma_0 \int_{-5/6}^{5/6} \tilde{f}^{+\prime}(x_1)^2 \varphi(x_1) \psi(\tilde{f}^{+}(x_1)) dx_1 \\
&+ \sigma_0 \int_{-5/6}^{5/6} \tilde{f}^{-\prime}(x_1)^2 \varphi(x_1) \psi(\tilde{f}^{-}(x_1)) dx_1 \\
&+ \sigma_0 \int_{-5/6}^{5/6} \int_{\tilde{f}^{-}(x_1)}^{\tilde{f}^{+}(x_1)} \tilde{f}^{+\prime\prime}(x_1) \varphi(x_1) \psi(x_2) dx_2 dx_1.
\end{aligned}
$$

This then implies that, for any $t > 0$ small,

$$
\begin{aligned}
&\lim_{\varepsilon \to 0} \frac{1}{h_\varepsilon^2} \int_{-5/6}^{5/6} \int_{f_\varepsilon^{-}(x_1)}^{f_\varepsilon^{-}(x_1)+th_\varepsilon} \left[\frac{1}{\varepsilon} W(u_\varepsilon) - \frac{\varepsilon}{2} \left| \frac{\partial u_\varepsilon}{\partial x_2} \right|^2 + \frac{\partial u_\varepsilon}{\partial x_1} \right|^2 \Bigg] \varphi(x_1) \\
= \; & \sigma_0 \int_{-5/6}^{5/6} \tilde{f}^{-\prime}(x_1)^2 \varphi(x_1) dx_1 + O(t).
\end{aligned}
$$

A similar identity holds on $\{f_\varepsilon^{+}(x_1) - th_\varepsilon < x_2 < f_\varepsilon^{+}(x_1)\}$.

Substituting the Modica inequality into this we obtain

$$
\lim_{\varepsilon \to 0} \frac{1}{h_\varepsilon^2} \int_{-5/6}^{5/6} \int_{f_\varepsilon^{-}(x_1)}^{f_\varepsilon^{-}(x_1)+th_\varepsilon} \varepsilon \left| \frac{\partial u_\varepsilon}{\partial x_1} \right|^2 \varphi(x_1) \le \sigma_0 \int_{-5/6}^{5/6} \tilde{f}^{-\prime}(x_1)^2 \varphi(x_1) dx_1 + O(t), \tag{15.32}
$$

and a similar inequality holds on $\{f_\varepsilon^{+}(x_1) - th_\varepsilon < x_2 < f_\varepsilon^{+}(x_1)\}$.

Because $h_\varepsilon \gg \varepsilon$, by (15.18) and [46, Lemma B.4],

$$
\lim_{\varepsilon \to 0} \frac{1}{h_\varepsilon^2} \int_{-5/6}^{5/6} \int_{f_\varepsilon^{-}(x_1)+th_\varepsilon}^{f_\varepsilon^{+}(x_1)-th_\varepsilon} \varepsilon \left| \frac{\partial u_\varepsilon}{\partial x_1} \right|^2 = 0. \tag{15.33}
$$

Combining (15.32) and (15.33), and noting that t can be arbitrarily small, we obtain

$$
\lim_{\varepsilon \to 0} \frac{1}{h_\varepsilon^2} \int_{-5/6}^{5/6} \int_{f_\varepsilon^{-}(x_1)}^{f_\varepsilon^{+}(x_1)} \varepsilon \left| \frac{\partial u_\varepsilon}{\partial x_1} \right|^2 \varphi(x_1) \le \sigma_0 \int_{-5/6}^{5/6} \left[\tilde{f}^{+\prime}(x_1)^2 + \tilde{f}^{-\prime}(x_1)^2 \right] \varphi(x_1) dx_1. \tag{15.34}
$$

Step 3. Combining the results in Step 1 and Step 2, we see

$$
\lim_{\varepsilon \to 0} \frac{1}{h_\varepsilon^2} \int_{-5/6}^{5/6} \int_{f_\varepsilon^{-}(x_1)}^{f_\varepsilon^{+}(x_1)} \varepsilon \left| \frac{\partial u_\varepsilon}{\partial x_1} \right|^2 \varphi(x_1) = \sigma_0 \int_{-5/6}^{5/6} \left[\tilde{f}^{+\prime}(x_1)^2 + \tilde{f}^{-\prime}(x_1)^2 \right] \varphi(x_1) dx_1, \tag{15.35}
$$

for any $\varphi \in C_0^\infty(-5/6, 5/6)$.

The conclusion of this lemma follows by substituting (15.28) and (15.29) into (15.35) and noting that φ is arbitrary. $\square$

A direct corollary is

Corollary 15.6. *Either $\tilde{f}^+(x_1) = -\tilde{f}^-(x_1) = \tau x_1 + \beta$ for some constant β, where $\tilde{f}^+ > 0$ strictly in $(-5/6, 5/6)$, or there exists a $\beta \in (-5/6, 5/6)$ such that $\tilde{f}^+(x_1) = -\tilde{f}^-(x_1) = \tau|x_1 - \beta|$.*

To finish the proof, it remains to exclude the later possibility. This will be done in the next subsection.

15.6 Completion of the proof

In this subsection we show that the one point contact of $\tilde{f}^+$ and $\tilde{f}^-$ is also impossible, hence complete the proof of Case 3 in the proof of Lemma 14.6.

By Corollary 15.6, after a translation and a scaling, we may assume that $\tilde{f}^+ = -\tilde{f}^- = \tau|x_1|$ for some constant $\tau > 0$.

Let

$$f_{\varepsilon,d}(x_1) := f_\varepsilon^+(x_1) - f_\varepsilon^-(x_1),$$

which is a positive convex function. Because

$$\lim_{\varepsilon \to 0} \frac{1}{h_\varepsilon} f_{\varepsilon,d}(x_1) = 2\tau|x_1|, \tag{15.36}$$

by the convexity, there exists a unique minimizer of $f_{\varepsilon,d}$ in $[-1,1]$, which is an interior point and converges to 0 as $\varepsilon \to 0$. For each ε, after a translation in the x_1-direction, we may assume the minima point of $f_{\varepsilon,d}$ is exactly the origin 0.

The following result can be viewed as the doubling property for stationary 2-valued harmonic functions (in the sense of Almgren) in dimension 1, see [50] for related results.

Lemma 15.7. *There exists a constant $c(\tau)$ depending only on τ such that, for any $r \in (-1/2, 1/2)$, if ε is small enough,*

$$f_{\varepsilon,d}\left(\frac{r}{2}\right) + f_{\varepsilon,d}\left(-\frac{r}{2}\right) \geq c(\tau)\left[f_{\varepsilon,d}(r) + f_{\varepsilon,d}(-r)\right]. \tag{15.37}$$

Proof. Take $c(\tau) := \min\{1/4, \tau/4\}$.

First note that, by (15.36), for any $\delta > 0$ fixed, if ε is small enough, (15.37) holds for any $r \in [\delta, 1/2)$.

We prove this lemma by contradiction. Hence assume by the contrary, there exists $r_\varepsilon \in (0, 1/4)$ such that for any $r \in [r_\varepsilon, 1/2)$,

$$\frac{f_{\varepsilon,d}\left(\frac{r}{2}\right) + f_{\varepsilon,d}\left(-\frac{r}{2}\right)}{f_{\varepsilon,d}(r) + f_{\varepsilon,d}(-r)} \geq c(\tau), \tag{15.38}$$

and

$$\frac{f_{\varepsilon,d}\left(\frac{r_\varepsilon}{2}\right)+f_{\varepsilon,d}\left(-\frac{r_\varepsilon}{2}\right)}{f_{\varepsilon,d}(r_\varepsilon)+f_{\varepsilon,d}(-r_\varepsilon)}=c(\tau). \tag{15.39}$$

In the below we will show these assumptions lead to a contradiction.

Because $c(\tau)<\tau/2$, we must have $r_\varepsilon\to 0$ as $\varepsilon\to 0$. Denote $\rho_\varepsilon:=f_{\varepsilon,d}(r_\varepsilon)+f_{\varepsilon,d}(-r_\varepsilon)$. Consider the scaling

$$v_\varepsilon(x_1,x_2):=u_\varepsilon(r_\varepsilon x_1,\rho_\varepsilon x_2),$$

and $\bar{f}_\varepsilon^\pm$ is defined as in the proof of Lemma 14.6. We also let

$$\bar{f}_{\varepsilon,d}:=\bar{f}_\varepsilon^+-\bar{f}_\varepsilon^-,$$

which is a positive convex function. By definition,

$$\bar{f}_{\varepsilon,d}(1)+\bar{f}_{\varepsilon,d}(-1)=1, \tag{15.40}$$

while by (15.38) and (15.39),

$$\bar{f}_{\varepsilon,d}(2)+\bar{f}_{\varepsilon,d}(-2)\le c(\tau)^{-1},\quad \bar{f}_{\varepsilon,d}(1/2)+\bar{f}_{\varepsilon,d}(-1/2)=c(\tau). \tag{15.41}$$

Hence $\bar{f}_{\varepsilon,d}$ is uniformly bounded on $[-2,2]$. Because it is convex and positive, we can assume it converges uniformly to a limit $\bar{f}_d$ on $[-3/2,3/2]$, which satisfies

$$\bar{f}_d(1)+\bar{f}_d(-1)=1, \tag{15.42}$$

and

$$\bar{f}_d(3/2)+\bar{f}_d(-3/2)\le c(\tau)^{-1},\quad \bar{f}_d(1/2)+\bar{f}_d(-1/2)=c(\tau). \tag{15.43}$$

Because $\rho_\varepsilon\gg\varepsilon$ (by Lemma 14.2), by (15.1),

$$\bar{f}_\varepsilon^+ + \bar{f}_\varepsilon^- = o(1). \tag{15.44}$$

Together with (15.41), this implies that both $\bar{f}_\varepsilon^\pm$ are uniformly bounded on $[-2,2]$. Hence by the convexity or concavity, they also converge uniformly to two limits $\bar{f}^\pm$ on $[-3/2,3/2]$, respectively.

By Corollary 15.6, $\bar{f}^\pm$ are piecewise linear. By (15.44), $\bar{f}^+=-\bar{f}^-$. Hence $\bar{f}_d=2\bar{f}^+\ge 0$.

If there is no contact point, $\bar{f}^+$ is linear. Then

$$\bar{f}_d(1)+\bar{f}_d(-1)=\bar{f}_d(1/2)+\bar{f}_d(-1/2).$$

This clearly contradicts (15.42) and (15.43) because $c(\tau)<1$.

Next assume there is a contact point, say $\bar{t}$. Then by Corollary 15.6, there exists a constant $\bar{\tau}>0$ such that

$$\bar{f}_d(x_1):=\bar{\tau}|x_1-\bar{t}|.$$

Here by (15.42), $\bar{\tau}$ and $\bar{t}$ should satisfy

$$\bar{\tau}|1-\bar{t}|+\bar{\tau}|1+\bar{t}|=1.$$

However, by direct calculation we obtain

$$\bar{f}_d(1/2)+\bar{f}_d(-1/2)=\bar{\tau}\left[\left|\frac{1}{2}-\bar{t}\right|+\left|\frac{1}{2}+\bar{t}\right|\right]\geq 1/2,$$

which contradicts (15.43) because $c(\tau)<1/2$. This finishes the proof. □

Denote $\rho_\varepsilon := f_{\varepsilon,d}(0)=f_\varepsilon^+(0)-f_\varepsilon^-(0)$. By our assumption, $\rho_\varepsilon \ll h_\varepsilon$. Since for any $r>0$,

$$\lim_{\varepsilon\to 0}\frac{f_{\varepsilon,d}(\pm r)}{h_\varepsilon}>0,$$

by continuity, there exists an $r_\varepsilon \to 0$ such that

$$f_{\varepsilon,d}(r_\varepsilon)+f_{\varepsilon,d}(-r_\varepsilon)=4\rho_\varepsilon. \tag{15.45}$$

Let $\epsilon := \varepsilon/r_\varepsilon$ and $u_\epsilon(x_1,x_2):=u_\varepsilon(r_\varepsilon x_1, r_\varepsilon x_2)$, which satisfies **B1)-B6)** with ε replaced by ϵ and $f_\varepsilon^\pm$ replaced by

$$f_\epsilon^\pm(x_1):=\frac{1}{r_\varepsilon}f_\varepsilon(r_\varepsilon x_1).$$

Proof of this claim. The situation is similar to the beginning of this section, except that we need to show $f_\epsilon^\pm$ converge to 0 uniformly. This follows from the following two facts.

1. By **B6)**, $f'_{\varepsilon,d}$ converges to 0 uniformly. Thus, recalling that 0 is the unique minimizer of $f_{\varepsilon,d}$, an integration gives

   $$2\rho_\varepsilon=\int_{-r_\varepsilon}^{r_\varepsilon}\left|f'_{\varepsilon,d}(x_1)\right|dx_1 \ll r_\varepsilon.$$

   Thus

   $$\sup_{x_1\in[-1,1]} f_{\epsilon,d}(x_1)\leq\frac{4\rho_\varepsilon}{r_\varepsilon}\to 0.$$

2. Note that we have performed a rotation so that $f_\varepsilon^+ + f_\varepsilon^- = O(\varepsilon)$. Thus, by noting that $r_\varepsilon \gg \rho_\varepsilon \gg \varepsilon$,

   $$f_\epsilon^+ + f_\epsilon^- = O(\varepsilon/r_\varepsilon)=O(\epsilon). \tag{15.46}$$

Combining these two points we see both $f_\epsilon^\pm$ converge to 0 uniformly. □

The above argument also shows that $r_\varepsilon \gg \rho_\varepsilon \gg \varepsilon$. Hence as $\varepsilon \to 0$, ϵ also goes to 0.

Similar to $f_{\varepsilon,d}$, denote $f_{\epsilon,d} := f_\epsilon^+ - f_\epsilon^-$.

By definition and the convexity of $f_{\epsilon,d}$, 0 is the unique minimizer of $f_{\epsilon,d}$. Moreover, by (15.45),

$$f_{\epsilon,d}(1) + f_{\epsilon,d}(-1) = 4f_{\epsilon,d}(0) = \frac{4\rho_\varepsilon}{r_\varepsilon}. \tag{15.47}$$

We can perform the same blow up procedure as before. Thus define

$$\bar{f}_\epsilon^\pm(x_1) := \frac{r_\varepsilon}{\rho_\varepsilon} f_\epsilon^\pm(x_1).$$

The blow up limit is denoted by $\bar{f}^\pm$. Denote $\bar{f}_{\epsilon,d} := \bar{f}_\epsilon^+ - \bar{f}_\epsilon^-$. Then by definition, $\bar{f}_{\epsilon,d}(0) = 1$. Clearly, 0 is still the unique minima point of $\bar{f}_{\epsilon,d}$. Thus $\bar{f}_{\epsilon,d} \geq 1$. Rescaling (15.47) gives

$$\bar{f}_{\epsilon,d}(1) + \bar{f}_{\epsilon,d}(-1) = 4. \tag{15.48}$$

Note that u_ϵ is in fact defined on $\mathcal{C}_2$. Thus the blow up sequence is defined on $(-2, 2)$. Moreover, by Lemma 15.7 and (15.47),

$$\bar{f}_{\epsilon,d}(2) + \bar{f}_{\epsilon,d}(-2) \leq 4c(\tau)^{-1}.$$

Combining this with (15.46), we see $\bar{f}_\epsilon^\pm$ are uniformly bounded on $[-2, 2]$. Hence we can assume they converge uniformly to $\bar{f}^\pm$ on $[-3/2, 3/2]$.

This uniform convergence allows us to passing to the limit in (15.48), which gives

$$\bar{f}_d(1) + \bar{f}_d(-1) = 4. \tag{15.49}$$

Similarly, $\bar{f}_d(0) = 1$ and $\bar{f}_d \geq 1$ in $[-3/2, 3/2]$. The latter fact, in combination with Corollary 15.6, implies that both $\bar{f}^\pm$ are linear. By (15.46), $\bar{f}^+ = -\bar{f}^-$. Therefore

$$\bar{f}^d = \bar{f}^+ - \bar{f}^- = 2\bar{f}^+ \tag{15.50}$$

is also linear. Since 0 is a minima point of $\bar{f}_d$ in $[-3/2, 3/2]$, this is possible only if $\bar{f}_d$ is a constant function. However, (15.49) contradicts with the fact that $\bar{f}_d(0) = 1$.

In conclusion, in Case 3 in the proof of Lemma 14.6, the blow up limit $\tilde{f}^+$ and $\tilde{f}^-$ cannot contact. Then by applying Corollary 15.6 again, we see both $\tilde{f}^\pm$ are linear functions. Coming back to the setting when we do not perform the rotation (which makes (15.1) holds), this says the blow up limit $\tilde{f}^- \equiv 0$, because it is linear and $\tilde{f}^-(0) = 0 \geq \tilde{f}^-(x_1)$ for any $x_1 \in (-1, 1)$. Then as before we get a contradiction and this finishes the proof of Case 3 in the proof of Lemma 14.6.

16 Exclusion of Case 4

In this section we consider Case 4 in the proof of Lemma 14.6. That is, we assume (notations as in the proof of Lemma 14.6) $\tilde{f}_d = +\infty$ everywhere in $(-1, 1)$.

By considering the functions $u_\varepsilon(r_\varepsilon x_1, r_\varepsilon x_2)$ and arguing as in Section 15, we are in the following situation:

1. u_ε is a sequence of solutions satisfying **B1)-B6)** (except that we may need to subtract a constant λ_ε so that $f_\varepsilon^+ - \lambda_\varepsilon$ converges and λ_ε may not converge to 0), with $\varepsilon \to 0$.

2. There exists $h_\varepsilon \to 0$ satisfying $h_\varepsilon \geq \lambda\varepsilon$ so that
$$\mathrm{osc}_{(-1,1)} f_\varepsilon^- = h_\varepsilon. \tag{16.1}$$

3. For any $x_1 \in (-1, 1)$,
$$\lim_{\varepsilon \to 0} \frac{f_\varepsilon^+(x_1) - f_\varepsilon^-(x_1)}{h_\varepsilon} = +\infty. \tag{16.2}$$

Because $f_\varepsilon^+ - f_\varepsilon^-$ is positive and convex, the convergence in (16.2) can be assumed to be uniform in $[-1, 1]$.

The main goal is still to prove that the blow up limit $\tilde{f}^- \equiv 0$ in $(-3/4, 3/4)$. This then leads to a contradiction as in Case 1-3.

16.1 The case $h_\varepsilon \gg \varepsilon$

Assume by the contrary, $\tilde{f}^-$ is not a linear function in $(-3/4, 3/4)$. Then by its concavity, there exists a point $t \in (-3/4, 3/4)$ and a linear function $\mathcal{L}$, which is a support line of $\tilde{f}^-$ at $(t, \tilde{f}^-(t))$, such that for some $a \in (0, 1/8)$, $\tilde{f}^- < \mathcal{L}$ strictly in $(t - a, t + a) \setminus \{t\}$.

After a rotation and a scaling, we are in the following settings: $\tilde{f}^-$ is a concave function satisfying $\tilde{f}^-(0) = 0$ and $\tilde{f}^- < 0$ strictly in $(-1, 1) \setminus \{0\}$. Take a constant $\lambda \in (0, \min\{-\tilde{f}^-(-1), -\tilde{f}^-(1)\})$. By sliding the function $x_2 = -\lambda x_1^2/2 + \text{const.}$ from above, we see there exists a constant t_λ such that the graph $x_2 = \widetilde{Q}_\lambda(x_1) := -\lambda x_1^2/2 + t_\lambda$ touches $\{x_2 = \tilde{f}^-(x_1)\}$ from above at an interior point in $(-1, 1)$. In fact, there exists a constant $\delta > 0$ such that
$$\widetilde{Q}_\lambda \geq \tilde{f}^- + \delta, \quad \text{in } [-1, -1 + \delta] \cup [1 - \delta, 1]. \tag{16.3}$$

$\tilde{f}^-$ is still obtained by blowing up another f_ε^- (associated to a solution u_ε, which satisfies all of the assumptions at the beginning of this section). Then for all ε small enough, there exists a constant $t_{\varepsilon,\lambda}$ such that the graph $\Gamma_\varepsilon := \{x_2 = Q_{\varepsilon,\lambda}(x_1) := -\frac{\lambda h_\varepsilon}{2} x_1^2 + t_{\varepsilon,\lambda} h_\varepsilon\}$ touches

$\{x_2 = f_\varepsilon^-(x_1)\}$ from above in $(-1,1)$. Moreover, the contact point lies in $[-1+\delta/2, 1-\delta/2]$. This is because, for all ε small, we have

$$Q_{\varepsilon,\lambda}(x_1) \geq f_\varepsilon^-(x_1) + \frac{\delta}{2}h_\varepsilon, \quad \text{in } [-1, -1+\frac{\delta}{2}] \cup [1-\frac{\delta}{2}, 1]. \tag{16.4}$$

Let $d_\varepsilon(x)$ be the distance of x to Γ_ε. Because Γ_ε is concave, this distance function is smooth above Γ_ε. In the following we denote

$$g_\varepsilon^*(x) := g\left(\frac{d_\varepsilon(x)}{\varepsilon}\right)$$

for $x \in \mathcal{C}_1$ lying above Γ_ε. We assume $g_\varepsilon^* = 0$ below Γ_ε.

Since d_ε is a convex function, we have

$$\varepsilon \Delta g_\varepsilon^* \geq \frac{1}{\varepsilon} W'(g_\varepsilon^*), \quad \text{in } \{g_\varepsilon^* > 0\}. \tag{16.5}$$

It also satisfies the free boundary condition $|\nabla g_\varepsilon^*| = \sqrt{2W(0)}/\varepsilon$ on $\partial\{g_\varepsilon^* > 0\}$.

For any $t > 0$, let

$$g_{\varepsilon,t}^*(x) := g_\varepsilon^*(x - te_2),$$

a translation of g_ε^* along the vertical direction.

Recall that $f_\varepsilon^+ - f_\varepsilon^- \gg h_\varepsilon$ in $(-1,1)$. Hence for any $L > 0$, if ε is small enough, $f_\varepsilon^+ - f_\varepsilon^- > 2Lh_\varepsilon$ in $(-1,1)$.

Lemma 16.1. *There exists an L_* independent of ε such that $u_\varepsilon > g_{\varepsilon,L_*h_\varepsilon}^*$ in $(-1,1) \times (L_*h_\varepsilon - 2h_\varepsilon, L_*h_\varepsilon + 2h_\varepsilon)$.*

Proof. By an ordinary differential equation analysis, we know

$$g(t) \leq 1 - ce^{-Ct}, \quad \text{in } (0, +\infty).$$

Hence in $(-1,1) \times (-2,2)$, by noting that $d_\varepsilon(x)$ is comparable to $x_2 - Q_{\varepsilon,\lambda}(x_1)$, we have

$$g_\varepsilon^*(x_1, x_2) \leq 1 - ce^{-C\frac{x_2 - Q_{\varepsilon,\lambda}(x_1)}{\varepsilon}}. \tag{16.6}$$

On the other hand, by the exponential convergence of u_ε away from free boundaries (14.10), in $(-1,1) \times (L_*h_\varepsilon - 2h_\varepsilon, L_*h_\varepsilon + 2h_\varepsilon)$,

$$u_\varepsilon(x_1, x_2) \geq 1 - Ce^{-cL_*\frac{h_\varepsilon}{\varepsilon}}.$$

The lemma follows by choosing L_* to satisfy $cL_* > 100C$. $\square$

Lemma 16.2. *For all ε small, $u_\varepsilon \geq g^*_\varepsilon$ on $\{x_1 = \pm 1 \mp \delta/4\} \times (-2h_\varepsilon, L_* h_\varepsilon)$. Moreover, this inequality is strict in $\{u_\varepsilon > 0\}$.*

Proof. Because $u_\varepsilon = g(\Psi_\varepsilon/\varepsilon)$, $g^*_\varepsilon = g(d_\varepsilon/\varepsilon)$, and g is increasing, we only need to show that $\Psi_\varepsilon > d_\varepsilon$ on $\{x_1 = \pm 1 \mp \delta/4\} \times (-2h_\varepsilon, L_* h_\varepsilon)$.

Let

$$\tilde{d}_\varepsilon(x_1, x_2) := \frac{1}{h_\varepsilon} d_\varepsilon(x_1, h_\varepsilon x_2).$$

It vanishes on $\widetilde{\Gamma}_\varepsilon := \{x_2 = \widetilde{Q}_{\varepsilon,\lambda}(x_1) := -\frac{\lambda}{2}x_1^2 + t_{\varepsilon,\lambda}\}$. Because $\widetilde{\Gamma}_\varepsilon$ touches $\{x_2 = \tilde{f}^-_\varepsilon(x_1)\}$ from the above and $\tilde{f}^-_\varepsilon$ converges to $\tilde{f}^-$ uniformly, we see $t_{\varepsilon,\lambda}$ converges to t_λ. Thus $\widetilde{Q}_{\varepsilon,\lambda}$ converges to $\widetilde{Q}_\lambda$ uniformly.

Because

$$\nabla d_\varepsilon(x_1, x_2) = \frac{(1, \lambda h_\varepsilon x_1)}{\sqrt{1 + \lambda^2 h_\varepsilon^2 x_1^2}},$$

$\frac{\partial \tilde{d}_\varepsilon}{\partial x_2}$ converges to 1 uniformly in $\{|x_1| < 1, \widetilde{Q}_{\varepsilon,\lambda}(x_1) < x_2 < L_*\}$. Then by the above analysis, $\tilde{d}_\varepsilon$ converges to $\left(x_2 - \widetilde{Q}_\lambda(x_1)\right)_+$ uniformly in $[-1 + \delta/8, 1 - \delta/8] \times [-2, L_*]$.

Recall that $\widetilde{\Psi}_\varepsilon(x_1, x_2)$ converges to $\left(x_2 - \tilde{f}^-(x_1)\right)_+$ uniformly in $[-1 + \delta/8, 1 - \delta/8] \times [-2, L_*]$. Then by (16.3), on $\{x_1 = \pm 1 \mp \delta/4\} \times [\widetilde{Q}^-(\pm 1 \mp \delta/4) - \delta/4, L_*]$,

$$\left(x_2 - \tilde{f}^-(x_1)\right)_+ \geq \left(x_2 - \widetilde{Q}_\lambda(x_1)\right)_+ + \frac{\delta}{4}.$$

Thus for all ε small, on $\{x_1 = \pm 1 \mp \delta/4\} \times [Q^-_\varepsilon(\pm 1 \mp \delta/4) - \frac{\delta h_\varepsilon}{4}, L_* h_\varepsilon]$,

$$\Psi_\varepsilon > d_\varepsilon.$$

On $\{x_1 = \pm 1 \mp \delta/4\} \times [f^-_\varepsilon(\pm 1 \mp \delta/4), Q^-_\varepsilon(\pm 1 \mp \delta/4) - \frac{\delta h_\varepsilon}{4}]$, we also have

$$\Psi_\varepsilon > 0 = d_\varepsilon.$$

In the remaining part, both Ψ_ε and d_ε are 0. ∎

Lemma 16.3. *$u_\varepsilon \geq g^*_\varepsilon$ in $(-1 + \delta/4, 1 - \delta/4) \times (-2, 2)$.*

Proof. For any $t > 0$, consider $g^*_{\varepsilon,t}$ introduced before. We want to show that for any $t \in [0, L_*]$, the following holds:
($\mathcal{P}_t$): $u_\varepsilon \geq g^*_{\varepsilon,t}$ in $\mathcal{D}_t := [-1 + \delta/4, 1 - \delta/4] \times [t h_\varepsilon - 2h_\varepsilon, t h_\varepsilon + 2h_\varepsilon]$.

By Lemma 16.1, this is true for $t = L_*$. Thus the following is well defined

$$t_* := \min\{t \in [0, L_*] : (\mathcal{P}_t) \text{ holds for all } t \in [t_*, L_*]\}.$$

Assume $t_* > 0$. By Lemma 16.3, for any $t > 0$, $u_\varepsilon > g^*_{\varepsilon,t}$ strictly on $\partial\mathcal{D}_t \cap \{x_1 = \pm(1-\delta/4)\}$. Because $\partial\{g^*_\varepsilon > 0\}$ lies below $\partial\{u_\varepsilon > 0\}$, $u_\varepsilon > 0 = g^*_{\varepsilon,t}$ on $\mathcal{D}_t \cap \partial\{g^*_{\varepsilon,t} > 0\}$. Using (16.5) and the strong maximum principle, we see that $u_\varepsilon > g^*_{\varepsilon,t_*}$ strictly in $\mathcal{D}_{t_*}$. Then by Lemma 16.3 and continuity, there exists a $\sigma > 0$ such that $(\mathcal{P}_t)$ holds for $t \in [t_* - \sigma, t_*]$. This is a contradiction with the definition of t_*. Thus we must have $t_* = 0$ and this lemma follows. $\square$

Since $u_\varepsilon \geq g^*_\varepsilon$, and $\partial\{u_\varepsilon > 0\}$ does not coincide with $\partial\{g^*_\varepsilon > 0\}$, by the strong maximum principle, we must have $u_\varepsilon > g^*_\varepsilon$ strictly in $\{g^*_\varepsilon > 0\}$. Let x_ε be the contact point of $\{x_2 = Q_{\varepsilon,\lambda}(x_1)\}$ and $\{x_2 = f^-_\varepsilon(x_1)\}$. By the convexity of $\partial\{g^*_\varepsilon > 0\}$, there exists a point $y_\varepsilon \in \{g^*_\varepsilon > 0\}$ and $r_\varepsilon \in (\delta h_\varepsilon/16, \delta h_\varepsilon/8)$ such that $B_{r_\varepsilon}(y_\varepsilon) \subset \{g^*_\varepsilon > 0\}$ and it touches $\partial\{g^*_\varepsilon > 0\}$ at x_ε.

By (16.5),

$$\Delta\left(u_\varepsilon - g^*_\varepsilon\right) \leq \frac{1}{\varepsilon^2}\frac{W'(u_\varepsilon) - W'(g^*_\varepsilon)}{u_\varepsilon - g^*_\varepsilon}\left(u_\varepsilon - g^*_\varepsilon\right).$$

Because $x_\varepsilon \in \partial\{u_\varepsilon > 0\} \cap \partial\{g^*_\varepsilon > 0\}$, $\left(u_\varepsilon - g^*_\varepsilon\right)(x_\varepsilon) = 0$. Then by Hopf Lemma,

$$\frac{\partial}{\partial\nu_\varepsilon}\left(u_\varepsilon - g^*_\varepsilon\right)(x_\varepsilon) > 0.$$

Here ν_ε is the normal vector of $\partial\{u_\varepsilon > 0\}$ (and also $\partial\{g^*_\varepsilon > 0\}$) at x_ε pointing to $\{u_\varepsilon > 0\}$. However, this is a contradiction with the fact that

$$\frac{\partial u_\varepsilon}{\partial\nu_\varepsilon}(x_\varepsilon) = |\nabla u_\varepsilon(x_\varepsilon)| = \frac{\partial g^*_\varepsilon}{\partial\nu_\varepsilon}(x_\varepsilon) = |\nabla g^*_\varepsilon(x_\varepsilon)| = \frac{\sqrt{2W(0)}}{\varepsilon}.$$

This finishes the proof of this case.

16.2 The case $h_\varepsilon \sim \varepsilon$

For simplicity, assume $h_\varepsilon = \varepsilon$. Let

$$v_\varepsilon(x_1, x_2) := u_\varepsilon(x_1, \varepsilon x_2),$$

and $\widetilde{\Psi}_\varepsilon$ defined by $v_\varepsilon = g(\widetilde{\Psi}_\varepsilon)$. $\widetilde{\Psi}_\varepsilon$ satisfies

$$\widetilde{\Psi}_\varepsilon(x_1, x_2) := \frac{1}{\varepsilon}\Psi(x_1, \varepsilon x_2). \tag{16.7}$$

By Lemma 14.3, for any $L > 0$,

$$\varepsilon\Big|\frac{\partial\widetilde{\Psi}_\varepsilon}{\partial x_1}\Big| + \Big|1 - \frac{\partial\widetilde{\Psi}_\varepsilon}{\partial x_2}\Big| \to 0, \tag{16.8}$$

uniformly on $\{(x_1, x_2) : |x_1| \le 4/5, \tilde{f}_\varepsilon^-(x_1) \le x_2 \le L\}$.

Because $\widetilde{\Psi}_\varepsilon = 0$ on $\{x_2 = \tilde{f}_\varepsilon^-(x_1)$ and $\tilde{f}_\varepsilon^-$ converges to $\tilde{f}^-$ uniformly on $[-4/5, 4/5]$, (16.8) implies that for any $L > 0$, $\widetilde{\Psi}_\varepsilon$ converges uniformly to $[x_2 - \tilde{f}^-(x_1)]_+$ uniformly on $[-4/5, 4/5] \times [-L, L]$. Then by (16.7), v_ε converges to $v = g(x_2 - \tilde{f}^-(x_1))$ uniformly on $[-4/5, 4/5] \times [-L, L]$ for any $L > 0$.

Recall that $\widetilde{V}_\varepsilon(x_1, x_2) := V_\varepsilon(x_1, \varepsilon x_2)/\varepsilon$ satisfies

$$\nabla^2 \widetilde{V}_\varepsilon = \begin{bmatrix} \frac{1}{\varepsilon^2}\left(W(v_\varepsilon) - \frac{1}{2}\left|\frac{\partial v_\varepsilon}{\partial x_2}\right|^2 + \frac{\varepsilon^2}{2}\left|\frac{\partial v_\varepsilon}{\partial x_1}\right|^2\right) & \frac{\partial v_\varepsilon}{\partial x_1}\frac{\partial v_\varepsilon}{\partial x_2} \\ \frac{\partial v_\varepsilon}{\partial x_1}\frac{\partial v_\varepsilon}{\partial x_2} & W(v_\varepsilon) + \frac{1}{2}\left|\frac{\partial v_\varepsilon}{\partial x_2}\right|^2 - \frac{\varepsilon^2}{2}\left|\frac{\partial u_\varepsilon}{\partial x_1}\right|^2 \end{bmatrix}.$$

As in Subsection 15.3, $\widetilde{V}_\varepsilon$ converges uniformly to a nonnegative convex function $\widetilde{V}$ on $[-4/5, 4/5] \times [-L, L]$ for any $L > 0$. Moreover,

$$\begin{aligned} & \int_{-4/5}^{4/5}\int_{\tilde{f}_\varepsilon^-(x_1)}^{L} \left|\frac{\partial v_\varepsilon}{\partial x_1}\right|^2 dx_2 dx_1 \\ \le\ & \int_{-4/5}^{4/5}\int_{\tilde{f}_\varepsilon^-(x_1)}^{L} \frac{1}{\varepsilon^2}\left(W(v_\varepsilon) - \frac{1}{2}\left|\frac{\partial v_\varepsilon}{\partial x_2}\right|^2 + \frac{\varepsilon^2}{2}\left|\frac{\partial v_\varepsilon}{\partial x_1}\right|^2\right) dx_2 dx_1 \\ =\ & \int_{-4/5}^{4/5}\int_{\tilde{f}_\varepsilon^-(x_1)}^{L} \frac{\partial^2 \widetilde{V}_\varepsilon}{\partial x_1^2} dx_2 dx_1 \\ \le\ & C(L). \end{aligned} \tag{16.9}$$

After passing to a subsequence, $\frac{\partial v_\varepsilon}{\partial x_1}$ converges weakly to $\frac{\partial v}{\partial x_1}$ in L^2. Then by the uniform convergence of $\frac{\partial v_\varepsilon}{\partial x_2}$, we get the weak L^2 convergence

$$\frac{\partial \widetilde{V}_\varepsilon}{\partial x_1 \partial x_2} \rightharpoonup \frac{\partial^2 \widetilde{V}}{\partial x_1 \partial x_2} = \frac{\partial v}{\partial x_1}\frac{\partial v}{\partial x_2}. \tag{16.10}$$

For any x_1 fixed and $x_2 \in [\tilde{f}_\varepsilon^-(x_1), L]$,

$$\frac{\partial^2 \widetilde{V}_\varepsilon}{\partial x_2^2}(x_1, x_2) = W(v_\varepsilon(x_1, x_2))\left[1 + \left|\frac{\partial \widetilde{\Psi}_\varepsilon}{\partial x_2}(x_1, x_2)\right|^2 - \frac{\varepsilon^2}{2}\left|\frac{\partial \widetilde{\Psi}_\varepsilon}{\partial x_1}(x_1, x_2)\right|^2\right], \tag{16.11}$$

which converges uniformly to $2W(v(x_1, x_2))$ (see Lemma 14.3). Hence for any $x_2 \in [\tilde{f}^-(x_1), L]$,

$$\frac{\partial^2 \widetilde{V}}{\partial x_2^2}(x_1, x_2) = 2W(v(x_1, x_2)) = g'(x_2 - \tilde{f}^-(x_1))^2. \tag{16.12}$$

By (16.11), $\widetilde{V}_\varepsilon(x_1,\cdot)$ are uniformly (with respect to ε and x_1) bounded in $C^{1,1}(-L,L)$. Hence $\widetilde{V}_\varepsilon(x_1,\cdot)$ converges to $\widetilde{V}(x_1,\cdot)$ in $C^1(-L,L)$. Because $\widetilde{V}_\varepsilon(x_1,\tilde{f}^-_\varepsilon(x_1)) = \frac{\partial \widetilde{V}_\varepsilon}{\partial x_2}(x_1,\tilde{f}^-_\varepsilon(x_1)) = 0$, $\widetilde{V}(x_1,\tilde{f}^-(x_1)) = \frac{\partial \widetilde{V}}{\partial x_2}(x_1,\tilde{f}^-(x_1)) = 0$. Then by (16.12), for any $t>0$,

$$\frac{\partial \widetilde{V}}{\partial x_2}(x_1,\tilde{f}^-(x_1)+t) = \int_0^t g'(s)^2 ds, \tag{16.13}$$

and

$$\widetilde{V}(x_1,\tilde{f}^-(x_1)+t) = \int_0^t \int_0^s g'(\tau)^2 d\tau. \tag{16.14}$$

In particular, $\widetilde{V}$ is a constant independent of x_1 on $\{x_2 = \tilde{f}^-(x_1)+t\}$. This then implies that

$$\begin{aligned}\frac{\partial \widetilde{V}}{\partial x_1}(x_1,\tilde{f}^-(x_1)+t) &= -\frac{\partial \widetilde{V}}{\partial x_2}(x_1,\tilde{f}^-(x_1)+t)\tilde{f}^{-\prime}(x_1) \\ &= -\tilde{f}^{-\prime}(x_1)\int_0^t g'(s)^2 ds. \end{aligned} \tag{16.15}$$

which holds for a.e. x_1. (Recall that $\widetilde{V}$ and $\tilde{f}$ are convex, thus Lipschitz and differentiable a.e. by Rademacher theorem.)

Differentiating (16.15) in x_1 once more, we obtain

$$\frac{\partial^2 \widetilde{V}}{\partial x_1^2}(x_1,\tilde{f}^-(x_1)+t) + \frac{\partial^2 \widetilde{V}}{\partial x_1 \partial x_2}(x_1,\tilde{f}^-(x_1)+t)\tilde{f}^{-\prime}(x_1) = -\tilde{f}^{-\prime\prime}(x_1)\int_0^t g'(s)^2 ds, \tag{16.16}$$

which is still understood in the a.e. sense (by using the a.e. second differentiability of convex functions, see [15].)

Substituting (see (16.10))

$$\begin{aligned}\frac{\partial^2 \widetilde{V}}{\partial x_1 \partial x_2}(x_1,\tilde{f}^-(x_1)+t)\tilde{f}^{-\prime}(x_1) &= \frac{\partial v}{\partial x_1}(x_1,\tilde{f}^-(x_1)+t)\frac{\partial v}{\partial x_2}(x_1,\tilde{f}^-(x_1)+t)\tilde{f}^{-\prime}(x_1) \\ &= -\Big|\frac{\partial v}{\partial x_1}(x_1,\tilde{f}^-(x_1)+t)\Big|^2, \end{aligned}$$

into (16.16) gives

$$\frac{\partial^2 \widetilde{V}}{\partial x_1^2}(x_1,\tilde{f}^-(x_1)+t) = \Big|\frac{\partial v}{\partial x_1}(x_1,\tilde{f}^-(x_1)+t)\Big|^2 - \tilde{f}^{-\prime\prime}(x_1)\int_0^t g'(s)^2 ds. \tag{16.17}$$

Next, for any $L > 0$, directly differentiating in x_1, using the equation for v_ε and integrating by parts leads to

$$
\begin{aligned}
& \frac{d}{dx_1} \int_{\tilde{f}_\varepsilon^-(x_1)}^{L} \frac{\partial v_\varepsilon}{\partial x_1}(x_1, x_2) \frac{\partial v_\varepsilon}{\partial x_2}(x_1, x_2) dx_2 \\
= \; & \frac{1}{\varepsilon^2} \left[W(v_\varepsilon(x_1, L)) - \frac{1}{2} \Big| \frac{\partial v_\varepsilon}{\partial x_2}(x_1, L) \Big|^2 + \frac{\varepsilon^2}{2} \Big| \frac{\partial v_\varepsilon}{\partial x_1}(x_1, L) \Big|^2 \right] \\
= \; & \frac{\partial^2 \widetilde{V}_\varepsilon}{\partial x_1^2}(x_1, L).
\end{aligned}
$$

For any nonnegative $\eta \in C_0^\infty(-4/5, 4/5)$, testing this with η and integrating by parts gives

$$
\int_{-4/5}^{4/5} \int_{\tilde{f}_\varepsilon^-(x_1)}^{L} \eta'(x_1) \frac{\partial v_\varepsilon}{\partial x_1}(x_1, x_2) \frac{\partial v_\varepsilon}{\partial x_2}(x_1, x_2) dx_2 dx_1 = - \int_{-4/5}^{4/5} \widetilde{V}_\varepsilon(x_1, L) \eta''(x_1) dx_1. \tag{16.18}
$$

By the uniform convergence of $\frac{\partial v_\varepsilon}{\partial x_2}$ and the L^2 weak convergence of $\frac{\partial v_\varepsilon}{\partial x_1}$, the left hand side of (16.18) converges to

$$
\begin{aligned}
& \int_{-4/5}^{4/5} \int_{\tilde{f}^-(x_1)}^{L} \eta'(x_1) \frac{\partial v}{\partial x_1}(x_1, x_2) \frac{\partial v}{\partial x_2}(x_1, x_2) dx_2 dx_1 \\
= \; & - \int_{-4/5}^{4/5} \eta'(x_1) \tilde{f}^{-\prime}(x_1) \int_{\tilde{f}^-(x_1)}^{L} g'(x_2 - \tilde{f}^-(x_1))^2 dx_2 dx_1,
\end{aligned} \tag{16.19}
$$

while by the uniform convergence of $\widetilde{V}_\varepsilon$, the right hand side converges to

$$
\begin{aligned}
- \int_{-4/5}^{4/5} \widetilde{V}(x_1, L) \eta''(x_1) dx_1 \; = \; & - \int_{-4/5}^{4/5} \frac{\partial^2 \widetilde{V}}{\partial x_1^2}(x_1, L) \eta(x_1) dx_1 \qquad (16.20) \\
= \; & \left[\int_0^{L - \tilde{f}^-(x_1)} g'(s)^2 ds \right] \left[\int_{-4/5}^{4/5} \tilde{f}(x_1) \eta''(x_1) dx_1 \right] \\
& - \int_{-4/5}^{4/5} \Big| \frac{\partial v}{\partial x_1}(x_1, L) \Big|^2 \eta(x_1) dx_1,
\end{aligned}
$$

where we have used (16.17).

Combining (16.18), (16.19) and (16.20) we get

$$
\int_{-4/5}^{4/5} \Big| \frac{\partial v}{\partial x_1}(x_1, L) \Big|^2 \eta(x_1) dx_1 = 0.
$$

Hence

$$0 = \frac{\partial v}{\partial x_1}(x_1, L) = -g'(L - \tilde{f}^-(x_1))\tilde{f}^{-\prime}(x_1).$$

Thus for all $x_1 \in (-4/5, 4/5)$, $\tilde{f}^-(x_1) = \tilde{f}^-(0) = 0$. This leads to a contradiction as before and finishes the proof of Lemma 14.6.

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